

THE MEDIAN CLASS AND SUPERRIGIDITY OF ACTIONS ON CAT(0) CUBE COMPLEXES

INDIRA CHATTERJI, TALIA FERNÓS, ALESSANDRA IOZZI

ABSTRACT. We define a bounded cohomology class, called the *median class*, in the second bounded cohomology – with appropriate coefficients – of the automorphism group of a finite dimensional CAT(0) cube complex X . This in turn defines a *median class of an action* by automorphisms of X .

We show that the median class of a non-elementary action by automorphisms does not vanish. We apply this result to establish a superrigidity result and show for example that no irreducible lattice in $\mathrm{SL}(2, \mathbb{R}) \times \mathrm{SL}(2, \mathbb{R})$, or more generally in the product of at least two locally compact groups with finitely many connected components, can act non-elementarily on a finite dimensional CAT(0) cube complex.

In the course of the proof we construct a Γ -equivariant measurable map from a Poisson boundary of Γ with values in the non-terminating ultrafilters on the Roller boundary of X .

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1. INTRODUCTION

The goal of this paper is to define a cohomological invariant of some non-positively curved metric spaces X for a non-elementary action of a group $\Gamma \rightarrow \mathrm{Aut}(X)$ and to use this invariant to establish rigidity phenomena.

The paradigm is that bounded cohomology with non-trivial coefficients is the appropriate framework to study negative curvature. The first instance of this fact is the Gromov–Sela cocycle on the real hyperbolic n -space X (in fact, on any simply connected space with pinched negative curvature) with values into the L^2 differential one-forms on X (see [Sel92] and [Gro93, 7.E₁]).

More recently, the same philosophy has been promoted by Monod [Mon06], Monod–Shalom [MS06, MS04] and Mineyev–Monod–Shalom [MMS04]. They prove that a non-elementary isometric action on a negatively curved space (belonging to a very rich class) yields the non-vanishing of second bounded cohomology with appropriately defined coefficients of a geometric nature. Such negatively curved spaces include proper CAT(-1) spaces, Gromov hyperbolic graphs of bounded valency, Gromov hyperbolic proper cocompact geodesic metric spaces, or simplicial trees.

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On the other hand, if G is a simple Lie group with rank at least two and \mathcal{H} is any unitary representation with no invariant vectors, then $H_{\text{cb}}^2(G, \mathcal{H}) = 0$, [BM99, BM02], thus showing that in non-positive curvature the situation cannot be expected to be completely analogous.

In this paper we move away from the negative curvature case and look at actions on CAT(0) cube complexes.

CAT(0) cube complexes are simply connected combinatorial objects introduced by Gromov in [Gro87]. They have been used in several important contexts, such as Moussong's characterization of word hyperbolic Coxeter groups in terms of their natural presentation, [Mou88]. A prominent use of CAT(0) cube complexes was made by Sageev in his thesis, [Sag95]: generalizing Stallings's theorem on the equivalence between splitting of groups and actions on trees, [Sco78, Ser74, Ser80], he proved an equivalence between the existence of an action of a group Γ on a CAT(0) cube complex and the existence of a subgroup $\Lambda < \Gamma$ such that the pair (Γ, Λ) has more than one end. More recently, Agol's proof of the last standing conjecture in 3-manifolds, the virtual Haken conjecture, uses (special) cube complexes in an essential way, thus indisputably asserting their relevance in the mathematical scenery.

The simplest example of a CAT(0) cube complex X is a simplicial tree; the midpoint of a vertex is the analogue of a *hyperplane* for a general CAT(0) cube complex. Hyperplanes separate X into two connected components, called *half spaces*, the collection of which is denoted by $\mathfrak{H}(X)$. If the vertex set of X is locally countable then $\mathfrak{H}(X)$ is countable as well. Two nested half spaces $h \subset k$ are *tightly nested* if for every half space ℓ such that $h \subseteq \ell \subseteq k$, either $\ell = h$ or $\ell = k$. In the case of a simplicial tree, tightly nested half spaces correspond to connected geodesic segments. We denote by $\mathfrak{H}(X)^{(n)}$ the collection of sequences of size $n \geq 1$ of *tightly nested* half spaces in $\mathfrak{H}(X)$.

A CAT(0) cube complex is in particular a *median space*, that is, given any three vertices, there is a unique vertex, the *median*, that is on the geodesics joining the three points. If $1 \leq p < \infty$, we define an $\text{Aut}(X)$ -invariant cocycle

$$c : X \times X \times X \rightarrow \ell^p(\mathfrak{H}(X)^{(n)})$$

as the sum of the characteristic functions of the finite subsets of the tightly nested half spaces “around” the median of three points, taken with the appropriate sign. Choosing a basepoint $v_0 \in X$ and evaluating c on an $\text{Aut}(X)$ -orbit, we get a cocycle on $\text{Aut}(X) \times \text{Aut}(X) \times \text{Aut}(X)$. It can be proven that, for every $n \geq 2$, the cocycle so defined is bounded and hence defines a bounded cohomology classes \mathfrak{m}_n , that we call a *median class*¹. (See (3.12), Proposition 3.3 and Lemma 3.4 for the precise definition and the proof of the above statements.)

If $\rho : \Gamma \rightarrow \text{Aut}(X)$ is an action of a group Γ by automorphisms on X , a *median class of the action* is the pull-back

$$\rho^*(\mathfrak{m}_n) \in H_{\text{b}}^2(\Gamma, \ell^p(\mathfrak{H}(X)^{(n)})).$$

¹For any $n \geq 2$ there is a median class, but in the following we will not necessarily make a distinction of the various median classes for different n .

Theorem 1.1. *Let X be a finite dimensional $CAT(0)$ cube complex with a Γ -action. If the Γ -action is non-elementary, then the median class of the Γ -action $\rho^*(\mathfrak{m}_n)$ does not vanish for all $n \geq 2$.*

We call an action $\Gamma \rightarrow \text{Aut}(X)$ *non-elementary* if there is no finite orbit in $X \sqcup \partial_{\triangleleft} X$, where $\partial_{\triangleleft} X$ denotes the visual boundary of X as a $CAT(0)$ space.

Let us say a word about what it means for a Γ -action to be non-elementary in the context of $CAT(0)$ cube complexes. First of all, the assumption implies in particular that, by passing to a subgroup of finite index, there are no Γ -fixed points in $\partial_{\triangleleft} X$. Under this hypothesis, using the work of Caprace-Sageev [CS11, Proposition 3.5], one can pass to a nonempty convex subset of X (called the Γ -essential core, see §2.4) which will have rather nice dynamic properties. Furthermore, the exclusion of a finite orbit on $\partial_{\triangleleft} X$ excludes the existence of a Euclidean factor in the essential core (see Corollary 2.18).

A key object in this paper is the *Roller boundary* ∂X of a $CAT(0)$ cube complex, defined in § 2.1. It arises naturally from considering its hyperplane (and hence half space) structure. The vertex set of X , together with its Roller boundary, can be thought of as a closed subset of a Bernoulli space (with $\mathfrak{H}(X)$ as the indexing set) and is hence totally disconnected. Although in the case of a tree the Roller boundary and the visual boundary coincide, we remark that in general, there is no natural map between them². Nevertheless, if the action is assumed to be essential (see § 2.4) and there is no finite orbit in $\partial_{\triangleleft} X$ then there is also no finite orbit in the Roller boundary³ ∂X .

One of the benefits of the Roller boundary is its robustness when considering products. Because of this, the cocycle can be defined for each irreducible factor of the essential core of X . We refer the reader to Proposition 3.14 for a description of the cocycle in the case in which the $CAT(0)$ cube complex is not irreducible. This, together with Theorem 1.1 yields immediately the following:

Corollary 1.2. *Let X be a finite dimensional $CAT(0)$ cube complex with a non-elementary action $\Gamma \rightarrow \text{Aut}(X)$. Then for all $n \geq 2$ and $1 \leq p < \infty$*

$$\dim H_b^2(\Gamma, \ell^p(\mathfrak{H}(X)^{(n)})) \geq m,$$

where $m \geq 1$ is the number of irreducible factors in the essential core of the Γ -action on X .

This result might not be sharp, in the sense that $H_b^2(\Gamma, \ell^p(\mathfrak{H}(X)^{(n)}))$ could be in some cases infinite dimensional.

Recently Bestvina–Bromberg–Fujiwara have announced the proof of the non-vanishing of the second bounded cohomology with general uniformly convex Banach spaces as coefficients and for weakly properly discontinuous actions on $CAT(0)$ spaces in the presence of a rank one isometry (that is an isometry whose axis does not bound a half flat).

²There is a map from the $CAT(0)$ boundary to a quotient of the Roller boundary [Gur], but we will not use it in this paper.

³This can be easily seen using the flipping lemma.

However, if the $\text{CAT}(0)$ cube complex is a product, then there are no rank one isometries. Caprace–Sageev proved [CS11] that there is always a decomposition of a $\text{CAT}(0)$ cube complex analogous to the decomposition of symmetric spaces into “irreducible” (or “rank one”) factors. Our result is not sensitive to this decomposition and hence also applies to products.

Moreover in Theorem 1.1, we are neither assuming that the action of $\Gamma \rightarrow \text{Aut}(X)$ is proper, nor that the $\text{CAT}(0)$ cube complex is proper or has a cocompact group of automorphisms.

Furthermore, our coefficients reflect geometric properties of the $\text{CAT}(0)$ cube complex, and this is essential to draw conclusions about the action. An example of this is the following superrigidity result:

Theorem 1.3 (Superrigidity). *Let Y be an irreducible finite dimensional $\text{CAT}(0)$ cube complex and $\Gamma < G_1 \times \cdots \times G_\ell =: G$ an irreducible lattice in the product of $\ell \geq 2$ locally compact groups. Let $\Gamma \rightarrow \text{Aut}(Y)$ be an essential and non-elementary action on Y . Then the action of Γ on Y extends continuously to an action of G , by factoring via one of the factors.*

Here the group $\text{Aut}(Y)$ is a topological group endowed with the topology of the pointwise convergence on vertices. This theorem is proven in § 6, to which we refer the reader also for an analogous result that does not require Y to be irreducible and the action to be essential.

We remark that requiring that the action is essential is necessary if one wants an irreducible $\text{CAT}(0)$ cube complex, as there is no guarantee that the essential core will be irreducible even when X is.

A result similar to Theorem 1.3 was proven by Monod [Mon06, Theorems 6 and 7] in the case of an infinite dimensional $\text{CAT}(0)$ space, with conditions both on the action and on the lattice Γ . For example, if Γ is not uniform, then in order to apply Monod’s version of Theorem 1.3, Γ has to be square-integrable and weakly cocompact. Although these conditions are verified for a large class of groups (such as for example Kazhdan Kac–Moody lattices and lattices in connected semisimple Lie groups), they are in general rather intractable. To give a sense of this, let us only remark that already finite generation (needed for example for square integrability) is not known for a lattice $\Gamma < \text{Aut}(\mathcal{T}_1) \times \text{Aut}(\mathcal{T}_2)$, not even by imposing strong conditions on the closure of the projections on Γ in $\text{Aut}(\mathcal{T}_i)$ to insure irreducibility.

Furthermore the more specific nature of a $\text{CAT}(0)$ cube complex versus a $\text{CAT}(0)$ space allows us to extend the action to the whole complex.

As an illustration we have the following immediate corollary that is a very special case of Theorem 1.3 (in fact of Corollary 6.2).

Corollary 1.4. *Let Γ be an irreducible lattice in $\text{SL}_2(\mathbb{R}) \times \text{SL}_2(\mathbb{R})$, then any Γ -action on a finite dimensional $\text{CAT}(0)$ cube complex is elementary.*

More generally, we have:

Corollary 1.5. *Let Γ be an irreducible lattice in the product $G := G_1 \times \cdots \times G_\ell$ of $\ell \geq 2$ locally compact groups with a finite number of connected components. Then any Γ -action on a finite dimensional CAT(0) cube complex is elementary.*

Indeed, since G has finitely many connected components and for any CAT(0) cube complex X the group $\text{Aut}(X)$ is totally disconnected, a continuous map from G to $\text{Aut}(X)$ must have finite image. The result applies in particular to semisimple Lie groups with finitely many components and real rank at least 2. Of course, it was already known that lattices in Lie groups with Property (T) cannot act non-elementarily on CAT(0) cube complexes [NR98]. Our result takes care of the remaining (non-simple) higher rank cases.

On a different tone, recall that the concept of measure equivalence was introduced by Gromov as a measure theoretical counterpart of quasi-isometries. The vanishing or non-vanishing of bounded cohomology is not invariant under quasi-isometries (see [BM99, Corollary 1.7]); on the other hand, Monod–Shalom proved that vanishing of bounded cohomology with coefficients in the regular representation is invariant under measure equivalence [MS06] and hence introduced a class of groups $\mathcal{C}_{reg} := \{\Gamma : H_b^2(\Gamma, \ell^2(\Gamma)) \neq 0\}$. They also proved for example that if $\Gamma \in \mathcal{C}_{reg}$ and $\Gamma \times \Gamma$ is measure equivalent to Λ , then $\Gamma \times \Gamma$ and Λ are commensurable. We can add to the groups in this list:

Corollary 1.6. *Let Γ be a group acting on a finite dimensional irreducible CAT(0) cube complex. If the action is metrically proper, non-elementary and essential, then $H_b^2(\Gamma, \ell^p(\Gamma)) \neq 0$, for $1 \leq p < \infty$, and hence in particular $\Gamma \in \mathcal{C}_{reg}$.*

We remark that the same result does not hold without the hypothesis of irreducibility (and hence essentiality). In fact, it can be easily seen using [BM02, Theorem 16], that if $\Gamma < G_1 \times G_2$ is an irreducible lattice in the product of locally compact groups, then $H_b^2(\Gamma, \ell^p(\Gamma)) = 0$, provided G_1 and G_2 are not compact. An example of such a group is any irreducible lattice Γ in $\text{SL}(2, \mathbb{Q}_p) \times \text{SL}(2, \mathbb{Q}_q)$, while it is easy to see that it acts non-elementarily and essentially on the product of two regular trees $\mathcal{T}_{p+1} \times \mathcal{T}_{q+1}$.

A result similar to Corollary 1.6 has been proven by Hamenstädt in the case of a group Γ acting properly on a proper CAT(0) space, also under the assumption that there exists a rank one isometry and that the group Γ is closed in the isometry group of X , [Ham]. Similarly, Hull and Osin proved that every group Γ with a sufficiently nice hyperbolic subgroup has infinite dimensional $H_b^2(\Gamma, \ell^p(\Gamma))$, for $1 \leq p < \infty$, [HO]. Examples of groups satisfying such condition encompass, among others, groups Γ acting properly on a proper CAT(0) space with a rank one isometry and groups Γ acting on a hyperbolic space also with a rank one isometry and containing a loxodromic element satisfying the Bestvina–Fujiwara “weakly properly discontinuous” condition. We emphasize that our CAT(0) cube complex are allowed to be locally countable and that, again, irreducibility is equivalent to the existence of a rank one isometry, [CS11].

The proof of Theorem 1.1 uses the functorial approach to bounded cohomology developed in [BM02, Mon01, BI02]; the main point here is to be able to realize bounded cohomology via

essentially bounded alternating cocycles on a Poisson boundary (B, ϑ) of Γ . The advantage is that the second bounded cohomology is not a quotient anymore (hence allowing to determine easily when a cocycle defines a non-trivial class); the disadvantage is that the pull-back via a representation has to be realized by a boundary map (with consequent technical difficulties, [BI02]). The amenability of the Poisson boundary implies immediately the existence of a boundary map into probability measures on the Roller compactification of X , but going from probability measures to Dirac masses is often the sore point of many rigidity questions. In the case of a proper CAT(0) cube complex and a cocompact group of isometries in $\text{Aut}(X)$, Nevo–Sageev identified the closure of the set of non-terminating ultrafilters (see § 2.1 for the definition) as a Poisson boundary of Γ . In this case, the boundary map could have been taken simply to be the identity. In general we have the following:

Theorem 1.7. *Let $\Gamma \rightarrow \text{Aut}(X)$ be a non-elementary group action on a finite dimensional CAT(0) cube complex X . If (B, ϑ) is a Poisson boundary for Γ , there exists a Γ -equivariant measurable map $\varphi : B \rightarrow \partial X$.*

In fact, one can obtain something a bit more precise, namely that the boundary map takes values in the non-terminating ultrafilters of the Γ -essential core of X (see Theorem 4.1 and Corollary 4.2). To prove Theorem 1.7, we develop some methods that take inspiration from [MS04, Proposition 3.3] in the case of a simplicial tree but are considerably more involved in the case of a CAT(0) cube complex due to the lack of hyperbolicity.

The first step in the identification of a Poisson boundary in [NS] is the proof that the set of non-terminating ultrafilters is not empty, under the assumption that the action is essential and the CAT(0) cube complex is cocompact. The same assertion with the cocompactness of X replaced by the non-existence of $\text{Aut}(X)$ -fixed points in the CAT(0) boundary follows from our proof that the boundary map takes values into the set of the non-terminating ultrafilters.

Corollary 1.8. *Let Y be a finite dimensional CAT(0) cube complex such that $\text{Aut}(Y)$ acts essentially and without fixed points in $\partial_{\infty} Y$. Then the set of non-terminating ultrafilters in ∂Y is not empty.*

The structure of the paper is as follows. In § 2 we recall the appropriate definitions and fix the terminology of CAT(0) cube complexes; we establish moreover some basic results needed in the paper, by pushing a bit further than what was available in the literature; the knowledgeable reader should have no problem skipping at least the first five subsections. In § 3 we construct the cocycle on the Roller compactification of the CAT(0) cube complex X and show that it is bounded. Moreover we give an explicit description of the cocycle on the essential core of the Γ -action in terms of the cocycles on its irreducible components. We conclude the section with an outlook on the proof of the non-vanishing of a median class. The boundary map and Theorem 1.7 are discussed in § 4. We prove Theorem 1.1 and Corollary 1.2 in § 5.2, while Theorem 1.3 and Corollary 1.6 are proven in § 6.

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2. PRELIMINARIES AND BASIC RESULTS

2.1. Generalities on CAT(0) Cube Complexes, Hyperplanes, Duality and Boundaries. A *cube complex* X is a metric polyhedral complex with cells isomorphic to $[0, 1]^n$ and isometries $\varphi_j : [0, 1]^j \rightarrow X$ as gluing maps. The cube complex is CAT(0) if it is non-positively curved with the induced Euclidean metric and has *finite dimension* D if the m -dimensional skeleton X^m of X is empty for $m > D$ and nonempty for $m = D$. We always assume our cube complex to be finite dimensional. A cube complex X is CAT(0) if and only if it is both simply connected and the link of every vertex is a flag complex: recall that a flag complex is a simplicial complex such that any $(n + 1)$ -vertices that are pairwise connected by an edge actually span an n -simplex, [BH99, Theorem II.5.20]. A *combinatorial isometry* between two CAT(0) cube complexes is a homeomorphism $f : X \rightarrow Y$ such that the composition $f \circ \varphi_j : [0, 1]^j \rightarrow Y$ is an isometry into a cube of Y . Note that any combinatorial isometry preserves also the CAT(0) metric. We denote by $\text{Aut}(X)$ the group of combinatorial isometries from X to itself.

Given a finite dimensional cube complex X , we can define an equivalence relation on edges, generated by the condition that two edges are equivalent if they are opposite sides of the same square (i.e. a 2-cube). A *midcube* of an n -cube σ with respect to the above equivalence relation is the convex hull of the set of midpoints of elements in the equivalence relation. A *hyperplane* is the union of the midcubes that intersect the edges in an equivalence class. So a hyperplane is a closed convex subspace and it defines uniquely two *half spaces*, that is the two complementary connected components. On the countable collection $\mathfrak{H}(X)$ – or simply \mathfrak{H} , when no confusion arises – of half spaces on X defined by the hyperplanes, one can define a fixed-point-free involution

$$(2.1) \quad \begin{aligned} * : \mathfrak{H} &\longrightarrow \mathfrak{H} \\ h &\mapsto h^* := X \setminus h, \end{aligned}$$

so that a hyperplane is the geometric realization of a pair $\{h, h^*\}$. In the following we identify the hyperplane \hat{h} with the pair of half spaces $\{h, h^*\}$ that it defines. We denote by $\hat{\mathfrak{H}}(X)$ the set of hyperplanes.

We say that two half spaces h, k are *transverse*, and we write $h \pitchfork k$, if all the intersections

$$(2.2) \quad h \cap k, \quad h \cap k^*, \quad h^* \cap k, \quad h^* \cap k^*$$

are not empty. Two half spaces h, k are *parallel*, and we write $h \parallel k$, if they are not transverse, equivalently if (exactly) one of the following relations

$$(2.3) \quad h \subset k^*, \quad h \subset k, \quad h^* \subset k^*, \quad h^* \subset k$$

holds; two parallel half spaces h and k are said to be *facing* if $h \subset k^*$. We say that two hyperplanes \hat{h}, \hat{k} are *transverse* (respectively *parallel*) if some (and hence any) choice of corresponding half spaces h and k is transverse (respectively parallel). Finally we say that two points u and v are *separated by* a half space h (or a hyperplane $\hat{h} = \{h, h^*\}$) if $u \in h$ and $v \in h^*$ (or vice-versa).

Two half-spaces h, k are said to be *nested* if either $h \subset k$ or $k \subset h$. Two nested half spaces $h \subset k$ are *tightly nested* if for every half space ℓ such that $h \subseteq \ell \subseteq k$, either $\ell = h$ or $\ell = k$. A subset of hyperplanes is *transverse* (respectively *parallel*) if all of its elements are pairwise transverse (respectively parallel).

Recall that a family of pairwise transverse hyperplanes must have a common intersection ([Sag95] or [Rolb]). We can think of the dimension of a CAT(0) cube complex as the largest cardinality of a family of pairwise transverse hyperplanes, because such a maximal intersection defines a cube of maximal dimension.

An *ultrafilter* α on \mathfrak{H} is a subset of \mathfrak{H} satisfying the following conditions:

- (i) For every $h \in \mathfrak{H}$ either $h \in \alpha$ or $h^* \in \alpha$.
- (ii) If $h \in \alpha$ and $k \supset h$, then $k \in \alpha$.

In other words, an ultrafilter on \mathfrak{H} is a choice of a half space for each hyperplane in X with the condition that as soon as a half space is in the ultrafilter, any half space containing it must also be in the ultrafilter. Equivalently, an ultrafilter is a section of the projection $\mathfrak{H} \rightarrow \hat{\mathfrak{H}}, h \mapsto \hat{h} = \{h, h^*\}$ satisfying the additional condition that for h, k in the image of that section, then it cannot happen that $h \subset k^*$. Those were called *admissible sections* in [CN05], but we will use the ultrafilter terminology⁴ since most authors seem to do so.

We say that an ultrafilter satisfies the *Descending Chain Condition (DCC)* if every descending chain of half spaces terminates. Such ultrafilters are called *principal* and are in one-to-one correspondence with the vertices of the CAT(0) cube complex X , [Gur]. By abuse of notation, we do not usually make a distinction between X , its vertex set, or the collection of principal ultrafilters.

The consideration of X as a collection of ultrafilters leads in a natural way to an inclusion of X into the Bernoulli space $2^{\mathfrak{H}}$, where $v \mapsto \{h \in \mathfrak{H} : v \in h\}$. This justifies a further (standard) abuse: thinking of $X \subset 2^{\mathfrak{H}}$, by duality we get that $h \in v$ if and only if $v \in h$. Let \overline{X} be the closure of X in $2^{\mathfrak{H}}$. One can check that the elements of \overline{X} , thought of as subsets of \mathfrak{H} , are ultrafilters.

⁴We point out that the notion of ultrafilter used in the theory of CAT(0) cube complexes is slightly off from the classical one in set theory and topology (see for example [CN74]). In fact, in the context of CAT(0) cube complexes, subsets of ultrafilters are never ultrafilters and thus, in particular, the intersection of two ultrafilters is never an ultrafilter.

The correspondence that associates to an ultrafilter a vertex in \overline{X} can be pushed further to give a duality between finite dimensional CAT(0) cube complexes and those *pocsets* that satisfy both the *finite interval condition* and the *finite width condition*. Recall that a *pocset* Σ is a partially ordered sets with an order reversing involution. The pocset satisfies the *finite interval condition* if for every pair $\alpha, \beta \in \Sigma$ with $\alpha \subset \beta$ there are only finitely many $\gamma \in \Sigma$ such that $\alpha \subset \gamma \subset \beta$; moreover if satisfies the *finite width condition* if there is an upper bound on the size of a collection of incomparable elements. Given a pocset Σ , one can consider the space of ultrafilters on Σ . The CAT(0) cube complex $X(\Sigma)$ corresponding to the pocset Σ has the principal ultrafilters as vertices, edges joining ultrafilters that differ only in the assignment on one element in Σ and cubes attached to the the 1-skeleton whenever it is possible.

The set of half spaces $\mathfrak{H}(X)$ in a CAT(0) cube complex is a pocset with the above properties and the CAT(0) cube complex obtained with the above construction from the set of principal ultrafilters on $\mathfrak{H}(X)$ is exactly X .

The boundary $\partial X := \overline{X} \setminus X$ is called the *Roller boundary*, and consists of all ultrafilters that are not principal, [Rola].

An ultrafilter $v \in \partial X$ is said to be *non-terminating* if every finite descending chain can be extended, i.e. if given a finite collection $\{h_0, \dots, h_N\} \subset v$ such that $h_0 \supset \dots \supset h_N$ there is an $h_{N+1} \in v$ such that $h_{N+1} \subset h_N \subset \dots \subset h_0$.

While the Roller boundary ∂X is not empty if the CAT(0) cube complex is unbounded, it is unclear as to when the set of non-terminating ultrafilters is not empty. However one can impose reasonable conditions which do guarantee that these exist (see [NS] and § 4). In the case of a tree, the entire Roller boundary consists of non-terminating ultrafilters; in the case of a \mathbb{Z}^D , there are only 2^D -many non-terminating ultrafilters, while there are examples, such as the wedge of two lines [NS, Remark 3.2], in which the set of non-terminating ultrafilter is empty.

A CAT(0) cube complex has of course also a visual boundary $\partial_{\text{q}} X$ with respect to its CAT(0) metric, [BH99]. We recall that $\partial_{\text{q}} X$ is the set of endpoints of geodesic rays in X , where we identify two geodesic rays if they stay at bounded distance from each other.

2.2. Intervals and Median Structure. For each $u, v \in X$ consider the oriented *interval* of half spaces

$$[u, v] = \{h \in \mathfrak{H} : h \in v \setminus u\}.$$

that is the (finite) set of half spaces containing v and not u . It is immediate to check that

$$[u, v] = [v, u]^*,$$

where $[v, u]^* := \{h \in \mathfrak{H} : h^* \in [v, u]\}$. The counting measure on \mathfrak{H} is consistent with the combinatorial metric d on X in that $|[u, v]| = d(u, v)$. We consider also the *vertex-interval*

$$\mathcal{I}(u, v) := \{w \in X : w \cap (u \cap v) = u \cap v\},$$

that is the set of all vertices that are crossed by some combinatorial geodesic between u and v .

The following fact seems to be folklore and is essential for our result. We refer to [BCG⁺09, Theorem 1.16] for a complete proof.

Lemma 2.1 (Intervals embedding). *Let $u, v \in \overline{X}$. Then the vertex intervals $\mathcal{I}(u, v)$ isometrically embed into $\overline{\mathbb{Z}^D}$ (with the standard cubulation) where D is the dimension of X .*

The image $\mathcal{I}_{u,v}$ in \mathbb{Z}^D of the above embedding is exactly the CAT(0) cube complex associated to the half spaces $\mathfrak{H}(u, v) := [u, v] \cup [v, u]$.

Remark 2.2. In general if $u \in X$ is an ultrafilter, its opposite $u^* := \{h \in \mathfrak{H}(X) : h^* \in u\}$ is not an ultrafilter. It is easy to see that if u^* is an ultrafilter, then $\mathfrak{H}(X) = [u, u^*] \cup [u^*, u]$ and hence X is an interval.

Also recall that the vertex set of a CAT(0) cube complex with the edge metric is a *median space* [Rola], namely for every triple of vertices $u, v, w \in X$, the intersection $\mathcal{I}(u, v) \cap \mathcal{I}(v, w) \cap \mathcal{I}(w, u)$ is exactly a singleton. This unique point is called the *median* of u, v , and w and we denote it by $m(u, v, w)$. It is a standard fact that

$$(2.4) \quad m = (u \cap v) \cup (v \cap w) \cup (w \cap u).$$

2.3. Decomposition into Products. As we alluded to after Lemma 2.1, if $\mathfrak{K} \subset \mathfrak{H}(X)$ is an involution invariant subset of hyperplanes, then \mathfrak{K} is a pocset in its own right and hence one can consider the associated CAT(0) cube complex $X(\mathfrak{K})$. A priori, the complex $X(\mathfrak{K})$ that one obtains with this construction will not be a subcomplex of X , but there is always a combinatorial quotient map $X \rightarrow X(\mathfrak{K})$, that is a projection with respect to the combinatorial metric. If the subset $\mathfrak{K} \subset \mathfrak{H}$ is invariant for the action of a group $\Gamma \rightarrow \text{Aut}(X)$ of combinatorial automorphisms, then the quotient map will be Γ -equivariant.

The product of CAT(0) cube complexes is a CAT(0) cube complex in a natural way. If $X = Y \times Z$, there is the following decomposition of the hyperplanes

$$(2.5) \quad \hat{\mathfrak{H}}(X) = \{\hat{h}_Y \times Z : \hat{h}_Y \in \hat{\mathfrak{H}}(Y)\} \sqcup \{Y \times \hat{h}_Z : \hat{h}_Z \in \hat{\mathfrak{H}}(Z)\} \cong \hat{\mathfrak{H}}(Y) \sqcup \hat{\mathfrak{H}}(Z),$$

and $(\hat{h}_Y \times Z) \pitchfork (Y \times \hat{h}_Z)$ for any $h_Y \in \mathfrak{H}(Y)$ and $h_Z \in \mathfrak{H}(Z)$.

Conversely, any such partition of the hyperplanes into mutually transverse subset corresponds to a decomposition of the CAT(0) cube complex into a product. In fact, by [CS11, Proposition 2.6] any CAT(0) cube complex decomposes as a product $X = X_1 \times \cdots \times X_m$ of irreducible factors, $m \geq 1$, which are unique up to permutations and are often referred to as the *rank one* factors of X .

The induced CAT(0) metric (respectively, the combinatorial metric) on the product is the ℓ^2 -product (respectively, ℓ^1 -product) of the factor metrics. We record the following standard fact:

Lemma 2.3. *Let $Y = Y_1 \times \cdots \times Y_k$ be the product of $CAT(0)$ spaces Y_j , $j = 1, \dots, k$ and let $G := G_1 \times \cdots \times G_k$, where $G_j := \text{Aut}(Y_j)$ is the group of isometries of the j -th factor Y_j . Then any G_j -fixed point in $\partial_{\triangleleft} Y_j$ defines a G -fixed point in $\partial_{\triangleleft} Y$.*

Proof. Let us denote by δ_j and δ the $CAT(0)$ metrics respectively on Y_j and on Y . Assume that, up to permuting the indices, there is a G_1 -fixed point $\xi_1 \in \partial_{\triangleleft} Y_1$. Let $\ell_1 : [0, \infty) \rightarrow Y_1$ be a geodesic in Y_1 representing ξ_1 , i.e. $\xi_1 = \ell_1(\infty)$. Since ξ_1 is G_1 -invariant, then $\sup_{t \in [0, \infty)} \delta_1(\gamma \ell_1(t), \ell_1(t)) < \infty$. If $y_j \in Y_j$ for $2 \leq j \leq m$ is any point, then $c : [0, \infty) \rightarrow Y$ defined by $\ell(t) := (c_1(t), y_2, \dots, y_m)$ is a geodesic in Y . Then for any $\gamma \in G$ we have

$$\sup_{t \in [0, \infty)} \delta_Y(\gamma \ell(t), \ell(t))^2 := \sup_{t \in [0, \infty)} \left[\delta_1(\gamma \ell_1(t), \ell_1(t))^2 + \sum_{j=2}^m \delta_j(\gamma y_j, y_j)^2 \right] < \infty,$$

hence $\ell(\infty)$ is G -invariant. ✱

In addition, there is a corresponding decomposition of the Roller boundary,

$$\partial X = \bigcup_{j=1}^m \overline{X_1} \times \cdots \times \overline{X_{j-1}} \times \partial X_j \times \overline{X_{j+1}} \times \cdots \times \overline{X_m},$$

and $\text{Aut}(X)$ contains $\text{Aut}(X_1) \times \cdots \times \text{Aut}(X_m)$ as a finite index subgroup ($\text{Aut}(X)$ is allowed to permute isomorphic factors). If $\Gamma \rightarrow \text{Aut}(X)$ is a group acting by automorphisms, then there is a subgroup $\Gamma_0 < \Gamma$ of finite index ($\leq m!$) that acts on X_j via the projection $\Gamma_0 \rightarrow \text{Aut}(X_j)$.

2.4. The Essential Core. A hyperplane $\hat{h} \in \hat{\mathfrak{H}}$ is called Γ -essential (or essential for short) if each of the corresponding half spaces contains points of a Γ -orbit arbitrarily far away from \hat{h} . The Γ -essential core (or essential core) Y of the Γ -action on X is a $CAT(0)$ cube complex corresponding to the essential hyperplanes. The Γ -action on Y is essential and any non-empty Γ -invariant convex subcomplex of Y is equal to Y . Following the notation of [CS11], we denote by $\text{Ess}(X, \Gamma)$ the set of Γ -essential hyperplanes in X , so that we can write

$$\hat{\mathfrak{H}}(X) = \text{Ess}(X, \Gamma) \sqcup \text{nEss}(X, \Gamma),$$

where the set of non-essential hyperplanes $\text{nEss}(X, \Gamma)$ includes both the half-essential and the trivial ones. What is important is that both $\text{Ess}(X, \Gamma)$ and $\text{nEss}(X, \Gamma)$ are Γ -invariant subsets of $\hat{\mathfrak{H}}(X)$ so

$$\hat{\mathfrak{H}}(X) = \text{Ess}(X, \Gamma) \sqcup \text{nEss}(X, \Gamma) = \text{Ess}(Y, \Gamma) \sqcup \text{nEss}(X, \Gamma),$$

While in general the essential core of an action can be empty, it is proven in [CS11, Proposition 3.5] that if there are no Γ -fixed points in the visual boundary $\partial_{\triangleleft} X$ of X and no Γ -fixed points in X , then the essential core Y is a non-empty Γ -invariant convex subcomplex $Y \subset X$. As a consequence, one has both that $\partial_{\triangleleft} Y \subset \partial_{\triangleleft} X$ and $\partial Y \subset \partial X$. However, even if X is irreducible, its essential core Y need not be. Let $Y = Y_1 \times \cdots \times Y_m$ be the decomposition into

its irreducible factors. Using the decomposition of hyperplanes for factors discussed above we obtain

$$(2.6) \quad \hat{\mathfrak{H}}(X) = \text{Ess}(Y, \Gamma) \sqcup \text{nEss}(X, \Gamma) = \hat{\mathfrak{H}}(Y_1) \sqcup \cdots \sqcup \hat{\mathfrak{H}}(Y_m) \sqcup \text{nEss}(X, \Gamma),$$

where we used for simplicity the notation $\hat{\mathfrak{H}}(Y_j)$ to indicate $\text{Ess}(Y_j, \Gamma)$ (since by hypothesis they coincide because the induced action on Y_j is Γ -essential).

Since if $s \in \hat{\mathfrak{H}}(X)^{(n)}$ is essential, then any other half space containing the half spaces in s is essential as well, the decomposition in (2.6) induces a decomposition

$$\hat{\mathfrak{H}}(X)^{(n)} = \hat{\mathfrak{H}}(Y_1)^{(n)} \sqcup \cdots \sqcup \hat{\mathfrak{H}}(Y_m)^{(n)} \sqcup \hat{\mathfrak{H}}_{\text{nEss}}(X)^{(n)},$$

where $\hat{\mathfrak{H}}_{\text{nEss}}(X)^{(n)}$ consists of tightly nested sequences at least one of which is non-essential.

2.5. Skewering, Flipping and Strongly Separated Hyperplanes. Flipping and double-skewering are important tools introduced by Caprace–Sageev in [CS11]

Definition 2.4 ([CS11]). We say that $\gamma \in \text{Aut}(X)$ *flips* a half-space $h \in \hat{\mathfrak{H}}(X)$ if $\gamma h^* \subset h$. Moreover we say that γ *skewers* \hat{h} if $\gamma h \subset h$ (or $h \subset \gamma h$).

Under reasonable hypotheses such combinatorial automorphisms can always be found. More precisely, if X is a finite dimensional CAT(0) cube complex and $\Gamma \rightarrow \text{Aut}(X)$ acts on X without fixing any point in the visual boundary $\partial_\infty X$, then for every half space $h \in \hat{\mathfrak{H}}(X)$, there exists $\gamma \in \Gamma$ that flips h , [CS11, Flipping Lemma, § 1.2]. As a simple consequence, we have also that, given any two half spaces $k \subset h$, there exists $\gamma \in \Gamma$ such that $\gamma h \subset k \subset h$, [CS11, Double Skewering Lemma, § 1.2].

The following notion, first introduced by Behrstock and Charney, [BC12], is essential in our application:

Definition 2.5 ([BC12]). We say that two hyperplanes are *strongly separated* if there is no hyperplane that is transverse to both.

By the usual abuse of terminology we say that two half spaces are strongly separated if the corresponding hyperplanes are.

The existence of strongly separated hyperplanes is definitively a rank one phenomenon. In fact, it is easy to see that if X is reducible, then there are no strongly separated hyperplanes. For non-elementary CAT(0) cube complexes, the fact that the existence of strongly separated hyperplanes is actually equivalent to the irreducibility of the CAT(0) cube complex was proven in [CS11], although the case of a right-angled Artin group can already be found in [BC12].

Lemma 2.6. *Let X be a finite dimensional irreducible CAT(0) cube complex and $\Gamma \subset \text{Aut}(X)$ a group acting essentially and non-elementarily. Given any hyperplane \hat{h} , there exists $\gamma \in \Gamma$ such that \hat{h} and $\gamma \hat{h}$ are strongly separated.*

Proof. By [CS11, Proposition 5.1] for any half space h there is a pair of half spaces h_1, h_2 such that $h_1 \subset h \subset h_2$ and \hat{h}_1 and \hat{h}_2 are strongly separated. We apply now the Double Skewering lemma in [CS11, § 1.2] to the pair $h_1 \subset h_2$ to obtain that $h_1 \subset h_2 \subset \gamma h_1$. By construction, and since Γ acts by automorphisms of X , we have the chain of inclusions

$$h_1 \subset h \subset h_2 \subset \gamma h_1 \subset \gamma h \subset \gamma h_2 \subset \gamma^2 h_1 \subset \gamma^2 h \subset \gamma^2 h_2.$$

Since \hat{h}_1 and \hat{h}_2 – and hence $\gamma \hat{h}_1$ and $\gamma \hat{h}_2$ – are strongly separated, it follows that \hat{h} and $\gamma^2 \hat{h}$ are strongly separated. \spadesuit

2.6. From Products to Irreducible Essential Factors. The following lemma identifies the important properties that are passed down from a complex to the irreducible factors of the essential core.

Lemma 2.7. *Let X be a finite dimensional CAT(0) cube complex and let $\Gamma \rightarrow \text{Aut}(X)$ be a non-elementary and essential action. Then the Γ -action on the irreducible factors of the essential core is also non-elementary and essential.*

Proof. Let $Y \subset X$ and let $Y = Y_1 \times \cdots \times Y_m$ be the decomposition of the essential core into irreducible factors. We need to show that the following hold:

- (1) The action Γ on the Y_j , $j = 1, \dots, m$ is essential as well.
- (2) If Γ has no finite orbit on the visual boundary $\partial_\triangleleft X$, then the same holds for the action on $\partial_\triangleleft Y_j$, $j = 1, \dots, m$.

(1) By [CS11, Proposition 3.2], the Γ -action on Y (resp. on Y_i) is essential if and only if every hyperplane $\hat{h} \in \hat{\mathfrak{H}}$ (resp. $\hat{h}_i \in \hat{\mathfrak{H}}_j$) can be skewed by some element in Γ . If $\hat{h}_j \in \hat{\mathfrak{H}}(Y_j)$ is a hyperplane in Y_j , then $\hat{h} := Y_1 \times \cdots \times Y_{j-1} \times \hat{h}_j \times Y_{j+1} \times \cdots \times Y_m$ is a hyperplane in Y . Since the action on Y is essential, there exists $\gamma \in \Gamma$ that skewers \hat{h} and hence it skewers \hat{h}_j . Then the Γ -action on Y_i is essential.

(2) We prove the contrapositive of the statement. Let $\Gamma_0 < \Gamma$ be the finite subgroup that preserves each of the factors Y_j and let us assume, by passing if necessary to a further subgroup of finite index, that there is a Γ_0 -fixed point in $\partial_\triangleleft Y_j$ for some $1 \leq j \leq m$. Then by Lemma 2.3 there is a Γ_0 -fixed point in $\partial_\triangleleft Y$ and hence a finite Γ -orbit in $\partial_\triangleleft Y$. Since Y is a convex subset of X and hence $\partial_\triangleleft Y \subset \partial_\triangleleft X$, there is a finite Γ -orbit in $\partial_\triangleleft X$. \spadesuit

2.7. Euclidean (Sub)Complexes.

Definition 2.8. Let X be a CAT(0) cube complex. We say that X is *Euclidean* if the vertex set with the combinatorial metric embeds isometrically in \mathbb{R}^D with the ℓ^1 -metric, for some $D < \infty$.

In [CS11, Theorem 7.2], under some natural conditions on the action of $\text{Aut}(X)$, the authors relate the existence of a $\text{Aut}(X)$ -invariant *Euclidean flat* with the non-existence of a facing triple of hyperplanes (that is a triple of hyperplanes associated to pairwise disjoint half

spaces). As our setting slightly differs from the one used in [CS11], we discuss briefly in this section the notion of Euclidean complexes and subcomplexes. The following definition is from [CS11].

Definition 2.9. A $\text{CAT}(0)$ cube complex X is said to be \mathbb{R} -like if there is an $\text{Aut}(X)$ -invariant bi-infinite $\text{CAT}(0)$ geodesic.

Proposition 2.10. *Let Y be a $\text{CAT}(0)$ cube complex on which $\text{Aut}(Y)$ acts essentially. Consider the following statements:*

- (1) Y is Euclidean.
- (2) Y is an interval.
- (3) Y is a product of \mathbb{R} -like factors.

Then $(3) \Rightarrow (2) \Rightarrow (1)$.

Proof. Observe that conditions (1) and (2) are preserved under taking products. Also, the hypothesis of having an essential action is preserved by passing to the irreducible factors by Lemma 2.7. Therefore, it is sufficient to consider the case in which Y is irreducible.

$(3) \Rightarrow (2)$. Assume that Y is \mathbb{R} -like. Let $\ell \subset Y$ be the $\text{Aut}(Y)$ -invariant $\text{CAT}(0)$ geodesic. We claim that ℓ crosses every hyperplane of Y . In fact, otherwise there would be a half space h_0 containing ℓ and, since ℓ is $\text{Aut}(Y)$ -invariant, then \hat{h}_0 would not be essential.

Let $\ell : \mathbb{R} \rightarrow Y$ be a parametrization of ℓ . One can check that, because of the above claim, the collection of half spaces

$$\alpha := \{h \in \mathfrak{H}(Y) : \text{there exists } t \in \mathbb{R} \text{ such that } h \supset \ell(t, \infty)\}$$

defines a non-terminating ultrafilter. Then Y is an interval on α and its opposite ultrafilter $\alpha^* = \mathfrak{H} \setminus \alpha$.

$(2) \Rightarrow (1)$ This is Lemma 2.1. ✱

We prove next that, under the assumption that there are no fixed points in the visual boundary and the action is essential, being Euclidean is equivalent to the non-existence of facing triples of hyperplanes⁵. As a byproduct, using [CS11, Theorem 7.2] we can conclude that also (1) implies (3) under the above hypotheses. We start with the following easy lemma.

Lemma 2.11. *If X is a Euclidean $\text{CAT}(0)$ cube complex that isometrically embed into \mathbb{R}^D , then any set of pairwise facing half spaces has cardinality at most $2D$.*

⁵It is possible that a Euclidean $\text{CAT}(0)$ cube complex Y on which $\text{Aut}(Y)$ acts essentially and without fixed points in the visual boundary, is a point (cf. [CS11, Theorem E]).

Proof. Indeed any collection of half spaces can be arranged in at most D chains. Hence for each dimension there can be at most one pair of facing half-spaces and the assertion follows from the fact that the ℓ^1 -metric on \mathbb{R}^D is the sum of the ℓ^1 -metrics on its factors. \clubsuit

More precisely, we have the following dichotomy that is compatible with the terminology in [CS11] but holds also in the case in which the $\text{CAT}(0)$ cube complex does not have a cocompact group of automorphisms.

Corollary 2.12. *Let Y be a finite dimensional irreducible $\text{CAT}(0)$ cube complex and assume that $\text{Aut}(Y)$ acts essentially and without fixed points on $\partial_{\triangleleft} Y$. Then Y is Euclidean if and only if $\mathfrak{H}(Y)$ does not contain a facing triple of half spaces.*

Proof. We first prove that if Y is Euclidean then there are no facing triples of half spaces. Since the action is essential and there are no fixed points in $\partial_{\triangleleft} Y$, if there is a facing triple of half spaces we can skewer several times two of the hyperplanes into the third half space to obtain a set of pairwise facing hyperplanes of arbitrarily large cardinality. Then Lemma 2.11 implies that Y is not Euclidean.

Conversely, we assume that there are no facing triples of hyperplanes and prove that Y must be Euclidean. Since Y is irreducible, let $\{h_n\}$ be a descending sequence of strongly separated half spaces, $h_{n+1} \subset h_n$. The strategy of the proof consists in showing that $\bigcap h_n$ consists of a single point $\alpha \in \partial Y$ and in using the non-existence of facing triples of hyperplanes to show that α^* is also an ultrafilter. Then Remark 2.2 will complete the proof.

To show that $\bigcap h_n$ is a single point, let us assume by contradiction that $\bigcap h_n$ contains at least two distinct points, $u, v \in \bigcap h_n$. Let \hat{h} be a hyperplane that separates them. Observe that for every $n \in \mathbb{N}$

$$(2.7) \quad \begin{aligned} u &\in h \cap h_n \neq \emptyset \text{ and} \\ v &\in h^* \cap h_n \neq \emptyset. \end{aligned}$$

From this and the fact that the h_n are a descending chain, one can check that, if there exists $N \in \mathbb{N}$ such that $\hat{h} \cap h_N$, then $\hat{h} \cap h_n$ for all $n \geq N$, which is impossible since the $\{h_n\}$ are pairwise strongly separated. So $\hat{h} \parallel \hat{h}_n$ for every $n \in \mathbb{N}$.

Again from (2.7) it follows that $\hat{h} \subset h_n$ for all $n \in \mathbb{N}$. But this is also not possible since there exist finitely many hyperplanes between \hat{h} and \hat{h}_n . Hence $\alpha := \bigcap h_n$ is a single point.

To see that α^* is an ultrafilter, we need only to check the admissibility condition, namely that if $h \in \alpha^*$ and $h \subset k$, then $k \in \alpha^*$. Observe that this is equivalent to verifying that if $\alpha \in h^*$ and $h \subset k$, then $\alpha \in k^*$. Suppose that this is not the case, that is that there exists $h, k \in \mathfrak{H}(Y)$ such that $h \subset k$ and $\alpha \in h^* \cap k$.

We first claim that

$$(2.8) \quad \text{there exists } n_0 \in \mathbb{N} \text{ such that } h_n \subset k \text{ for all } n > n_0.$$

In fact, suppose that there exists $n'_0 \in \mathbb{N}$ such that $\hat{h}_{n'_0} \pitchfork \hat{k}$. Since the $\{h_n\}$ are pairwise strongly separated, then $\hat{h}_n \parallel \hat{k}$ for all $n > n'_0$. Using (2.2), the fact that the $\{h_n\}$ are a descending chain and that $\alpha \in h_n$ for all $n \in \mathbb{N}$, it is easy to verify that $h_n \subset k$ for all $n > n'_0$.

On the other hand, if $\hat{h}_n \parallel \hat{k}$ for all $n \in \mathbb{N}$, using again that the $\{h_n\}$ are a descending chain and that there are only finitely many hyperplanes between any \hat{h}_n and \hat{k} , one can easily verify that there exists $n''_0 \in \mathbb{N}$ such that $h_n \subset k$ for all $n > n''_0$. Hence (2.8) is verified with $n_0 = \max\{n'_0, n''_0\}$.

Since $h \subset k$ and there are only finitely many hyperplanes between \hat{h} and \hat{k} , there exists $n_1 \geq n_0$ such that either $\hat{h}_{n_1} \pitchfork \hat{h}$ or $h_{n_1} \subset h$. But h_{n_1} cannot be contained in h since $\alpha \in h_{n_1} \cap h^*$, hence $\hat{h}_{n_1} \pitchfork \hat{h}$.

Again because the hyperplanes $\{\hat{h}_n\}$ are strongly separated, if $n > n_1$ then $\hat{h}_n \parallel \hat{h}$. This, the fact that $\hat{h}_{n_1} \pitchfork \hat{h}$ and that $h_n \subset h_{n_1}$ imply that $h_n \cap h = \emptyset$.

It follows that h_n, h and k^* is a facing triple of half spaces, contradicting the hypothesis. Hence α^* is an ultrafilter and the proof is complete. \clubsuit

2.8. n -Disjoint Facing Triples of Half Spaces. In the first part of this section we show how the hypotheses of non-elementarity and essentiality of the action are used to construct a facing triple of hyperplanes. This is standard, and is recalled here because the argument needs to be pushed a bit further to obtain Lemma 2.17. Furthermore Lemmas 2.14 and 2.13 are used in Corollary 2.18 to exclude the presence of Euclidean factors.

Lemma 2.13. *Let Y be a $CAT(0)$ cube complex with an action $\Gamma \rightarrow \text{Aut}(Y)$ without fixed points on $\partial_\triangleleft Y$. If all hyperplanes are compact, then any hyperplane is part of a facing triple, \hat{h}, \hat{k} and \hat{w} .*

Proof. Let $\gamma_0 \in \Gamma$ be a hyperbolic element with axis ℓ and let \hat{h} be a hyperplane crossed by ℓ . The hyperplanes $\gamma_0^{-m}\hat{h}, \gamma_0^m\hat{h}$ are also crossed by ℓ and, using compactness of \hat{h} , they are parallel for some $m \in \mathbb{N}$ large enough. Since there are no fixed points in $\partial_\triangleleft X$, there is an element $\gamma_1 \in \Gamma$ such that ℓ and $\gamma_1\ell$ are not at bounded distance from each other. For $n \in \mathbb{N}$ large enough one can check that $\gamma_1\gamma_0^n\gamma_1^{-1}\hat{h}, \gamma_0^{-m}\hat{h}, \gamma_0^m\hat{h}$ is a facing triple of hyperplanes. \clubsuit

The following lemma is part of the proof of [CS11, Theorem 7.2].

Lemma 2.14. *Let Y be an irreducible $CAT(0)$ cube complex and let $\Gamma \rightarrow \text{Aut}(Y)$ be a group that acts essentially on Y and without fixed points on $\partial_\triangleleft Y$. Then any non-compact hyperplane $\hat{h} \in \hat{\mathfrak{H}}(Y)$ is part of a facing triple of hyperplanes, \hat{h}, \hat{k} and \hat{w} .*

Proof. Since Y is irreducible, we may apply [CS11, Proposition 5.1] to find a pair of strongly separated hyperplanes \hat{h}', \hat{h}'' such that $h' \subset h \subset h''$. By [CS11, Double Skewering Lemma,

§ 1.2] there is some $g \in \Gamma$ that double skewers \hat{h}' past \hat{h}'' , with $h'' \subset gh'$. Hence

$$g^{-1}h \subset g^{-1}h'' \subset h' \subset h \subset h'' \subset gh' \subset gh,$$

and in particular $g^{-2}\hat{h}$, \hat{h} and $g^2\hat{h}$ are pairwise strongly separated.

Since \hat{h} is not compact, it crosses infinitely many hyperplanes. once none of them can cross both $g^{-2}\hat{h}$ and $g^2\hat{h}$ and only finitely many of them can separate $g^{-2}\hat{h}$ and $g^2\hat{h}$, it follows that there is a half space k that crosses \hat{h} and contains both $g^{-2}\hat{h}$ and $g^2\hat{h}$. Then $g^{-2}\hat{h}$, \hat{h} and $g^2\hat{h}$ form a facing triple of hyperplanes. By applying g^2 to the triple, we obtain that also \hat{h} , $g^2\hat{h}$ and $g^4\hat{h}$ is a facing triple of hyperplanes. \spadesuit

The above two lemmas together show that if there are no fixed points in the visual boundary, there are facing triple of hyperplanes. As mentioned above, we need to refine this result to show that there are facing triples in any Γ -orbit and that the pairwise facing hyperplanes are “far enough” from each other. We start with the following:

Definition 2.15. Two facing half spaces h_1, h_2 are n -disjoint if there exists a sequence of distinct tightly nested half spaces such that

$$h_1 \subset k_1 \cdots \subset k_\ell \subset h_2^*$$

for some $\ell \geq n$.

We remark that if two half spaces are n -disjoint, then they are k -disjoint for all $k \leq n$ and, in particular, they are facing.

Definition 2.16. An n -disjoint facing triple of half spaces is a triple of pairwise n -disjoint half spaces.

We deduce the following:

Lemma 2.17. *Let Y be a finite dimensional CAT(0) cube complex. Let $\Gamma \rightarrow \text{Aut}(Y)$ be a group that acts essentially on Y , with no finite orbit on the visual boundary of Y . Then for every half space $h \in \mathfrak{H}(Y)$, and every n there exist $\gamma, \gamma' \in \Gamma$ such that the half spaces $h, \gamma h, \gamma' h$ are an n -disjoint facing triple. If furthermore Y is irreducible, then they can be made pairwise strongly separated as well.*

Proof. Let $\Gamma_0 < \Gamma$ be the finite index subgroup that leaves invariant each irreducible factor Y_j of Y . By passing if necessary to a further finite index subgroup, we may assume that Γ_0 acts essentially on each irreducible factor Y_j and without fixed points in the visual boundary $\partial_\infty Y_j$, $j = 1, \dots, m$. Since by the decomposition in (2.5) a facing triple in an irreducible factor gives rise to a facing triple in Y , it is enough to consider the case in which Y is irreducible.

Let Y be irreducible. It will be enough to find a facing triple in any Γ -orbit in $\mathfrak{H}(Y)$. In fact, if necessary, we may continue to skewer the hyperplanes until they are n -disjoint and pairwise strongly separated.

By Lemma 2.14, any non-compact hyperplane $\hat{h} \in \hat{\mathfrak{H}}(Y)$ is part of a facing triple \hat{h}, \hat{k} and \hat{w} . By double skewering \hat{h} past \hat{k} and past \hat{w} we obtain a facing triple in the Γ -orbit of \hat{h} .

If, in addition to the non-compact \hat{h} , there were to exist also a non-compact hyperplane $\hat{h}_0 \in \hat{\mathfrak{H}}(Y)$ one could equally skewer \hat{h}_0 past \hat{k} and past \hat{w} to obtain a facing triple in the Γ -orbit of \hat{h}_0 .

Finally if all hyperplanes were compact, then any compact hyperplane \hat{h}_0 could be skewered into any facing triple whose existence follows from Lemma 2.13 to get a facing triple in the Γ -orbit of \hat{h}_0 . \clubsuit

We conclude the section with the following corollary that will be paramount in the following.

Corollary 2.18. *Let X be a finite dimensional CAT(0) cube complex and $\Gamma \rightarrow \text{Aut}(X)$ a non-elementary action. Then there are no Euclidean factors in the essential core.*

Proof. Let $Y \subset X$ be the essential core of the Γ -action and let Y_0 be an irreducible factor of Y . By Lemma 2.7 the Γ -action on Y_0 is also essential and non-elementary. By Lemma 2.17 there are facing triples of hyperplanes and hence, by Corollary 2.12, Y_0 cannot be Euclidean. \clubsuit

3. CONSTRUCTION AND BOUNDEDNESS OF THE MEDIAN CLASS

Let Γ be a group and E be a coefficient Γ -module, that is the dual of a separable Banach space on which Γ acts by linear isometries. The bounded cohomology of Γ with coefficients in E is the cohomology of the subcomplex of Γ -invariants in $(C_b(\Gamma^{k+1}, E), d)$, where

$$(3.9) \quad C_b(\Gamma^k, E) := \{f : \Gamma^k \rightarrow E : \sup_{g \in \Gamma^k} \|f(g)\|_E < \infty\},$$

is endowed with the Γ -action

$$(gf)(g_1, \dots, g_k) := g \cdot f(g^{-1}g_1, \dots, g^{-1}g_k),$$

and

$$d : C_b(\Gamma^k, E) \longrightarrow C_b(\Gamma^{k+1}, E)$$

is the usual homogeneous coboundary operator defined by

$$df(g_0, \dots, g_k) := \sum_{j=0}^k (-1)^j f(g_0, \dots, g_{j-1}, g_{j+1}, \dots, g_k).$$

3.1. The Median Cocycle on \overline{X} . Let X be a finite dimensional CAT(0) cube complex and $\mathfrak{H}(X)^{(n)}$ the collection of sequences of length n of tightly nested half spaces. Let \overline{X} be the Roller compactification of X , that is the set of ultrafilters on $\mathfrak{H}(X)$.

If $1 \leq p < \infty$, then $\ell^p(\mathfrak{H}(X)^{(n)})$ is the dual of a separable Banach space. In fact, if $1 < p < \infty$, then $\ell^p(\mathfrak{H}(X)^{(n)})$ is the dual of $\ell^q(\mathfrak{H}(X)^{(n)})$, where $1/p + 1/q = 1$. On the other hand,

$\ell^1(\mathfrak{H}(X)^{(n)})$ is the dual of the Banach space $C_0(\mathfrak{H}(X)^{(n)})$ of functions on $\mathfrak{H}(X)^{(n)}$ that vanish at infinity, which is separable since $\mathfrak{H}(X)^{(n)}$ is countable. For further use, we set the notation

$$(3.10) \quad \mathcal{E}_p := \begin{cases} \ell^q(\mathfrak{H}(X)^{(n)}) & 1 < p < \infty \text{ and } 1/p + 1/q = 1 \\ C_0(\mathfrak{H}(X)^{(n)}) & p = 1 \end{cases}$$

We define in this section a cocycle

$$c^{(n)} : \overline{X} \times \overline{X} \times \overline{X} \longrightarrow \ell^p(\mathfrak{H}(X)^{(n)}),$$

that, by evaluation on a basepoint in \overline{X} will give a cocycle on $\Gamma \times \Gamma \times \Gamma$. We define the cocycle c as the coboundary of an unbounded $\text{Aut}(X)$ -invariant map $\omega^{(n)}$ on $\overline{X} \times \overline{X}$ and we will show that, on the other hand, $c^{(n)} = d\omega^{(n)}$ is bounded in the sense of (3.9). For $n \geq 2$, the *median class* \mathfrak{m}_n will be defined as the cohomology class of $c^{(n)}$ (independent of the basepoint).

If $u, v \in \overline{X}$, we set

$$[[u, v]]^{(n)} = \{s \in \mathfrak{H}(X)^{(n)} : s \subset v \setminus u\}.$$

So, $[[u, v]]^{(n)}$ is the collection of sequences of length n of nested consecutive half-spaces that contain v and not u , hence $s = (h_1, \dots, h_m) \in [[u, v]]^{(n)}$ implies in particular that $h_1 \supset \dots \supset h_n$. We hope that the notation suggests that these are in some sense, sub-intervals.

We will simply write c , ω and $[[u, v]]$ for $c^{(n)}$, $\omega^{(n)}$ and $[[u, v]]^{(n)}$ when the context is clear.

We now consider the supremum norm on \mathbb{Z}^D and if $u, v \in \overline{X}$, we denote by $d_\infty(u, v)$ the ℓ^∞ diameter of this isometric embedding $\mathcal{I}(u, v) \hookrightarrow \overline{\mathbb{Z}^D}$.

Lemma 3.1. *Let $n \in \mathbb{N}$. If $d_\infty(u, v) > n$ then there are at least $d_\infty(u, v) - (n - 1)$ many elements in $[[u, v]]$.*

Proof. By definition of the ℓ^∞ -norm, there is some coordinate in the embedding of the vertex interval $\mathcal{I}(u, v)$ which has diameter $d_\infty(u, v)$, which means that there are this many parallel hyperplanes at least in that direction. Then there are $d_\infty(u, v) - (n - 1)$ ways to choose n consecutive half spaces in that direction. \clubsuit

For $u, v \in \overline{X}$, let us define

$$(3.11) \quad \omega_{u,v} := \mathbb{1}_{[[u, v]]} - \mathbb{1}_{[[v, u]]}.$$

Fixing $u, v \in \overline{X}$, the function has finitely many values

$$\omega_{u,v} : \mathfrak{H}(X)^{(n)} \rightarrow \{-1, 0, 1\}$$

and is only finitely supported if $u, v \in X$. However, also in this case ω is not bounded when thought of as a function with values in $\ell^p(\mathfrak{H}(X)^{(n)})$. In fact, we have the following

Lemma 3.2. *The map $(u, v) \mapsto \omega_{u,v}$ is $\text{Aut}(X)$ -equivariant. Moreover*

$$\|\omega_{u,v}\|_p^p \geq 2(d_\infty(u, v) - n + 1)^p.$$

Proof. The equivariance of ω is straightforward from the definitions. To see the other assertion, observe that for every $u, v, w \in \overline{X}$ we have that $[[u, v]] \cap [[v, w]] = \emptyset$. In particular, $[[u, v]] \cap [[v, u]] = \emptyset$ so that $\|\omega_{u,v}\|_p^p = \|\mathbb{1}_{[[u, v]]}\|_p^p + \|\mathbb{1}_{[[v, u]]}\|_p^p$. Then Lemma 3.1 concludes the proof. \clubsuit

Let us now consider the $\text{Aut}(X)$ -equivariant cocycle taking values in the functions on $\mathfrak{H}(X)^{(n)}$, defined as

$$\begin{aligned} (3.12) \quad c(u_1, u_2, u_3) &:= (d\omega)_{u_1, u_2, u_3} \\ &= \omega_{u_2, u_3} - \omega_{u_1, u_3} + \omega_{u_1, u_2} = \omega_{u_2, u_3} + \omega_{u_3, u_1} + \omega_{u_1, u_2} \\ &= \mathbb{1}_{[[u_2, u_3]]} + \mathbb{1}_{[[u_3, u_1]]} + \mathbb{1}_{[[u_1, u_2]]} - (\mathbb{1}_{[[u_3, u_2]]} + \mathbb{1}_{[[u_1, u_3]]} + \mathbb{1}_{[[u_2, u_1]]}) . \end{aligned}$$

We will show that, contrary to ω , the cocycle c on \overline{X} actually takes values in $\ell^p(\mathfrak{H}(X)^{(n)})$ (Proposition 3.3) and is bounded in the sense of (3.9).

3.2. Boundedness of the Median Class.

Proposition 3.3. *Let X be a finite dimensional $\text{CAT}(0)$ cube complex and let c be the cocycle defined in (3.12). Then*

$$(3.13) \quad c : \overline{X} \times \overline{X} \times \overline{X} \longrightarrow \ell^p(\mathfrak{H}(X)^{(n)})$$

and

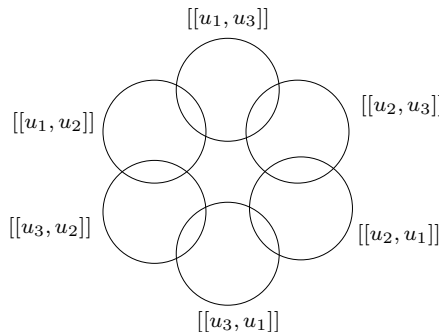
$$\sup_{u_1, u_2, u_3 \in \overline{X}} \|c(u_1, u_2, u_3)\|_p < \infty .$$

More precisely, for any $u_1, u_2, u_3 \in \overline{X}$, the support of $c(u_1, u_2, u_3)$ has cardinality bounded by $6(n-1)D^n$, where D is the dimension of X .

Proof. Let us first examine the structure of the intersections of the six sets appearing in the definition of c . Observe that for $a, b, i, j \in \{1, 2, 3\}$ and $a \neq b$ and $i \neq j$ we have that

$$\text{if } a \neq i \text{ and } b \neq j, \text{ then } [[u_a, u_b]] \cap [[u_i, u_j]] = \emptyset .$$

This is described more clearly by the following diagram:



Indeed, consider $s \in [[u_a, u_b]] \cap [[u_i, u_a]]$. Then every $h \in s$ must contain u_b and not u_a but must also contain u_a and not u_i , which shows that the intersection is empty. Likewise, $[[u_a, u_b]] \cap [[u_b, u_j]] = \emptyset$.

However c vanishes on each of the pairwise non-empty intersections. Indeed, if $s \in [[u_1, u_2]] \cap [[u_1, u_3]]$, then

$$c(u_1, u_2, u_3)(s) = 0 + 0 + \mathbb{1}_{[[u_1, u_2]]}(s) - (0 + \mathbb{1}_{[[u_1, u_3]]}(s) + 0) = 0.$$

The other cases are computed similarly. For further use, we collect the information obtained so far.

Lemma 3.4. *The support of $c(u_1, u_2, u_3)$ is the disjoint union of the six sets obtained by permuting the indices of $[[u_1, u_3]] \setminus ([[u_1, u_2]] \cup [[u_2, u_3]])$. On each of these sets the cocycle is identically equal to 1 or -1 .*

We are hence left to compute the cardinality of the set

$$[[u_1, u_3]]^{(n)} \setminus ([[u_1, u_2]]^{(n)} \cup [[u_2, u_3]]^{(n)}) =: S^{(n)}(u_1, u_3; u_2)$$

(in this part of the proof it will be convenient to keep track of the length of the sequences).

To this purpose, let $s = (h_1, \dots, h_n) \in S^{(n)}(u_1, u_3; u_2)$. Then there exists $j = 1, \dots, n-1$ such that $u_2 \in h_j$ and $u_2 \notin h_{j+1}$. Hence we can write s as the concatenation $s = s^{(j)} \bullet s^{(n-j)}$ of the two subsequences $s^{(j)}$ and $s^{(n-j)}$, where

$$\begin{aligned} s^{(j)} &= (h_1, \dots, h_j) \in [[u_1, u_3]]^{(j)} \cap [[u_1, u_2]]^{(j)} \\ s^{(n-j)} &= (h_{j+1}, \dots, h_n) \in [[u_1, u_3]]^{(n-j)} \cap [[u_2, u_3]]^{(n-j)}. \end{aligned}$$

Using (2.4) to compute the median $m = m(u, v, w)$ of any three points $u, v, w \in \overline{X}$, an easy verification shows that

$$[u, w] \cap [v, w] = [m, w] \quad \text{and} \quad [u, w] \cap [u, v] = [u, m].$$

In other words, the hyperplanes that separate u and v from w are exactly those that separate the median from w ; likewise, the hyperplanes that separate u from v and w are precisely those that separate u from the median m . Hence we can write

$$\begin{aligned} h \in s^{(j)} &\Rightarrow h \in [u_1, m] \quad \text{and} \\ h \in s^{(n-j)} &\Rightarrow h \in [m, u_3]. \end{aligned}$$

It follows that

$$S^{(n)}(u_1, u_3; u_2) = \bigcup_{j=1}^{n-1} ([u_1, m]]^{(j)} \bullet [[m, u_3]]^{(n-j)} \cap [[u_1, u_3]]^{(n)},$$

where $[[u_1, m]]^{(j)} \bullet [[m, u_3]]^{(n-j)}$ denotes the concatenation of a sequence in $[[u_1, m]]^{(j)}$ with a sequence in $[[m, u_3]]^{(n-j)}$ to obtain a tightly nested sequence in $[[u_1, u_3]]^{(n)}$.

The idea is to estimate from above the cardinality of the sets on the right hand side. The fact that the concatenation of a sequence in $[[u_1, m]]^{(j)}$ with a sequence in $[[m, u_3]]^{(n-j)}$ must be a tightly nested sequence in $[[u_1, u_3]]^{(n)}$ corresponds exactly to two properties: (1) the last half space h_j in the sequence in $[[u_1, m]]^{(j)}$ must be parallel to the first half space h_{j+1} in the sequence in $[[m, u_3]]^{(n-j)}$, and (2) the two half spaces h_j and h_{j+1} must cross two cubes that intersect (at least) in the median. Notice moreover that these two cubes are uniquely determined by the fact that they are contained respectively in the interval from u_1 to m and m to u_3 .

To estimate the cardinality of $S^{(n)}(u_1, u_3; u_2)$ we may as well release the condition (1), as already only (2) will give us a finite upper bound.

We denote by $[[u_1, m]]_0$ the tightly nested sequences obtained by fixing the ending cube and by ${}_0[[m, u_3]]$ the tightly nested sequences obtained by fixing the starting cube, so that

$$S^{(n)}(u_1, u_3; u_2) \subseteq \bigcup_{j=1}^{n-1} ([u_1, m]]_0^{(j)} \bullet {}_0[[m, u_3]]^{(n-j)}.$$

Since X has dimension D , intervals of \overline{X} embed in $\overline{\mathbb{Z}^D}$, once we fix a half-space there are only at most D choices for the next nested one. It follows that

$$|[u_1, m]]_0^{(j)}| \leq D^j \quad \text{and} \quad |{}_0[[m, u_3]]^{(n-j)}| \leq D^{n-j}.$$

and hence $|S^{(n)}(u_1, u_3; u_2)| \leq (n-1)D^n$. ✱

3.3. The Cocycle in Terms of its Essential Irreducible Components. If $Y \subset X$ is the essential core of the Γ -action on X and $Y = Y_1 \times \cdots \times Y_m$ is the decomposition of Y into irreducible $\text{CAT}(0)$ cube complexes, then one has both the cocycle c defined on \overline{X} and with values in $\ell^p(\mathfrak{H}(X)^{(n)})$ and the cocycles c_j defined on $\overline{Y_j}$ and with values in $\ell^p(\mathfrak{H}(Y_j)^{(n)})$. It is natural to ask what is the relation between these cocycles. In fact, from the decomposition in (2.5) it follows that we have a corresponding decomposition

$$(3.14) \quad \ell^p(\mathfrak{H}(X)^{(n)}) \cong \ell^p(\mathfrak{H}(Y_1)^{(n)}) \oplus \cdots \oplus \ell^p(\mathfrak{H}(Y_m)^{(n)}) \oplus \ell^p(\mathfrak{H}_{\text{Ess}}(X)^{(n)}),$$

given by $f \mapsto \mathbb{1}_{\mathfrak{H}(Y_1)^{(n)}} f + \cdots + \mathbb{1}_{\mathfrak{H}(Y_m)^{(n)}} f + \mathbb{1}_{\mathfrak{H}_{\text{Ess}}(X)^{(n)}} f$, where the direct sum is in the ℓ^p sense. The direct summand $\ell^p(\mathfrak{H}(Y_j)^{(n)})$ are invariant for the action of a finite index subgroup $\Gamma' < \Gamma$. We denote by $\pi_j : \overline{Y} \rightarrow \overline{Y_j}$ the projection.

Proposition 3.5. *Let $Y = Y_1 \times \cdots \times Y_m$ be the decomposition into irreducible factors of the essential core Y of the finite dimensional $\text{CAT}(0)$ cube complex X . For $1 \leq j \leq m$, let us denote by*

$$c_j : \overline{Y_j} \times \overline{Y_j} \times \overline{Y_j} \rightarrow \ell^p(\mathfrak{H}(Y_j)^{(n)})$$

the cocycle on the irreducible factors and let c be the cocycle on

$$c : \overline{Y} \times \overline{Y} \times \overline{Y} \rightarrow \ell^p(\mathfrak{H}(Y)^{(n)}).$$

Then c can be decomposed as

$$c(\xi, \eta, \zeta) = c_1(\pi_1(\xi), \pi_1(\eta), \pi_1(\zeta)) \oplus \cdots \oplus c_m(\pi_m(\xi), \pi_m(\eta), \pi_m(\zeta)).$$

In particular if $c_j(\xi_j, \eta_j, \zeta_j) \neq 0$, for some $1 \leq j \leq m$, then $c(\xi, \eta, \zeta) \neq 0$ for every $(\xi, \eta, \zeta) \in \pi_j^{-1}(\xi_j, \eta_j, \zeta_j)$.

Proof. Let ω and ω_j , for $j = 1, \dots, m$ be defined as in (3.11) respectively on \overline{Y} and \overline{Y}_j . Since $c_j = d\omega_j$ for $1 \leq j \leq k$ and $c = d\omega$, it is enough to verify that

$$\omega = \omega_1 + \cdots + \omega_m.$$

Let $(\xi, \eta) \in \overline{Y} \times \overline{Y}$ and set $\xi_j := \pi_j(\xi)$ for $1 \leq j \leq m$. Since $\omega_j(\xi_j, \eta_j) = \mathbf{1}_{[[\xi_j, \eta_j]]} - \mathbf{1}_{[[\eta_j, \xi_j]]}$ and $\omega(\xi, \eta) = \mathbf{1}_{[[\xi, \eta]]} - \mathbf{1}_{[[\eta, \xi]]}$, it is enough to see that

$$[[\xi, \eta]] = [[\xi_1, \eta_1]]_1 \sqcup \cdots \sqcup [[\xi_m, \eta_m]]_m,$$

where $[[\cdot, \cdot]]_j \subset \mathfrak{H}(Y_j)^{(n)}$. But this follows immediately from the structure of the hyperplanes and half spaces in a product. \clubsuit

3.4. How to Prove that the Median Class Does not Vanish. We defined at the beginning of this section the bounded cohomology of Γ with coefficients in $\ell^p(\mathfrak{H}(X)^{(n)})$ as the cohomology of the complex of the Γ -equivariant bounded functions on the Cartesian product Γ^k with values in $\ell^p(\mathfrak{H}(X)^{(n)})$. So far we constructed a Γ -equivariant cocycle $c : \overline{X} \times \overline{X} \times \overline{X} \rightarrow \ell^p(\mathfrak{H}^{(n)})$ and we remarked that a choice of a basepoint will give a cocycle in $C_b(\Gamma^3, \ell^p(\mathfrak{H}(X)^{(n)}))$. We still need to show that the cohomology class represented by this cocycle does not vanish. In order to do this, we recall from [BM02, Mon01] that, if (B, ϑ) is a Poisson boundary for Γ , there is an isometric isomorphism

$$(3.15) \quad H_b^2(\Gamma, \ell^p(\mathfrak{H}(X)^{(n)})) \cong \mathcal{Z}L_{\text{alt},*}^\infty(B^3, \ell^p(\mathfrak{H}(X)^{(n)}))^\Gamma,$$

where the space on the right hand side is the space of L^∞ alternating Γ -equivariant cocycles on $B \times B \times B$, with the measurability intended with respect to the weak-* topology on $\ell^p(\mathfrak{H}^{(n)}(X))$, $1 \leq p < \infty$ as a dual of \mathcal{E}_p (see (3.10)).

We recall that, in our specific case, the essential properties of the Γ -action on the Poisson boundary (B, ϑ) are the following:

- (1) it is amenable, and
- (2) any weak-* measurable Γ -equivariant map $B \times B \rightarrow \ell^p(\mathfrak{H}^{(n)}(X))$ is essentially constant.

One of the advantages of this realization is that, in degree two, there are no coboundaries, and hence showing that a cohomology class does not vanish amounts simply to showing that the corresponding cocycle is non-zero. The disadvantage is that realizing a bounded cohomology class on the boundary is possible under the condition that there exists a Γ -equivariant measurable boundary map $\varphi : B \rightarrow \overline{X}$ and that the bounded cohomology class can be represented by a Borel measurable alternating cocycle, [BI02].

It is immediate to verify that the cocycle c defined in this section is alternating in (u_1, u_2, u_3) , that is to say that if σ is a permutation of $\{u_1, u_2, u_3\}$ then

$$(3.16) \quad c(\sigma(u_1, u_2, u_3)) = \text{sign}(\sigma)c(u_1, u_2, u_3).$$

The Borel measurability of c is proven in Lemma A.4.

Furthermore, a Poisson boundary exists for any locally compact and compactly generated group according to [BM02], and for arbitrary locally compact groups with respect to a symmetric measure ϑ according to [Kai03]; the existence of the boundary map will take up the next section.

4. THE BOUNDARY MAP

This section is devoted to the proof of the following theorem, with an eye to the implementation of the isomorphism in (3.15).

Theorem 4.1. *Let $\Gamma \rightarrow \text{Aut}(Y)$ be a group action on an irreducible finite dimensional CAT(0) cube complex Y . Assume the action is essential and non-elementary. If B is a Poisson boundary of Γ , there exists a Γ -equivariant measurable map $\varphi : B \rightarrow \partial Y$ taking values into the non-terminating ultrafilters in ∂Y .*

To realize the isomorphism in (3.15) in our generality, we will in fact need the following stronger statement which guarantees the existence of some kind of boundary map when the action is not assumed to be essential.

Corollary 4.2. *Let $\Gamma \rightarrow \text{Aut}(X)$ be a group acting on a finite dimensional CAT(0) cube complex X . Assume that there is no finite orbit in the visual boundary $\partial_{\triangleleft} X$ and denote by Y the essential core of X . Then there exists a Γ -equivariant measurable map $\varphi : B \rightarrow \partial X$.*

Proof. Since the action of Γ has no finite orbits in $\partial_{\triangleleft} X$, in particular it has no fixed points. Therefore, the essential core Y is not empty, [CS11, Proposition 3.5], and Γ also has no finite orbit in $\partial_{\triangleleft} Y$ as well. If $Y = Y_1 \times \cdots \times Y_m$ is the decomposition of Y into a product of irreducible subcomplexes, by Lemma 2.7, Γ also has no finite orbit in $\partial_{\triangleleft} Y_i$, for $i = 1, \dots, m$ and moreover the action on each Y_i is essential.

If $j = 1, \dots, q$, let $\varphi_j : B \rightarrow \partial Y_j$ be the Γ -equivariant measurable boundary map whose existence is proven in Theorem 4.1. Since $\prod_{j=1}^q \partial Y_j \subseteq \partial Y \subseteq \partial X$, the map $\varphi : B \rightarrow \partial Y$ defined by $\varphi(b) := (\varphi_1(b), \dots, \varphi_q(b))$ has the desired properties. \clubsuit

The idea of the proof of Theorem 4.1 is as follows. Since \overline{X} is a continuous compact metric G -space, the space $\mathcal{P}(\overline{X})$ of probability measures on \overline{X} endowed with the weak-* topology is a subset of the (unit ball in the) dual of the continuous functions on \overline{X} . By amenability of the Γ -action on B , there exists a Γ -equivariant measurable map $\psi : B \rightarrow \mathcal{P}(\overline{X})$ into the probability measures on \overline{X} (see [Zim84, Proposition 4.3.9]). Each probability measure μ on \overline{X} divides the set of half spaces into “balanced” (that is half spaces such that $\mu(h) = \mu(h^*)$)

and “unbalanced” ones. If all half spaces are unbalanced this defines an ultrafilter, hence the map $\psi : B \rightarrow \mathcal{P}(\overline{X})$ gives a Γ -equivariant map $\psi : B \rightarrow \overline{X}$. Since the measure ϑ on B is ergodic so is the push-forward measure on \overline{X} . Hence up to measure 0 the image of ψ is either in X or in ∂X . If it is in X then it is essentially constant so we get a Γ -fixed point, hence it had to land in ∂X . The whole work in the proof will be to exclude the presence of balanced half spaces using non-elementary actions assumptions as well as essentiality.

4.1. General Preliminary Lemmas Using Ergodicity. The following lemma can be thought of as a weaker version of the statement that a Poisson boundary for a lattice is a Poisson boundary for its ambient group and vice versa.

Lemma 4.3. *Let Γ be a group acting on a measure space (M, ϑ) . If Γ acts ergodically on $(M \times M, \vartheta \times \vartheta)$, then every finite index subgroup $\Gamma_0 \leq \Gamma$ acts ergodically on (M, ϑ) .*

Proof. Let $\Gamma_0 \leq \Gamma$ be of finite index which does not act ergodically and let $M_0 \subset M$ be a Γ_0 -invariant subset which is neither null or co-null. Consider the subset of the product $\bigcup_{[\gamma] \in \Gamma/\Gamma_0} \gamma M_0 \times \gamma M_0$ (which is well defined by the Γ_0 -invariance of M_0). This set is Γ -invariant, it has measure $[\Gamma : \Gamma_0] \cdot \vartheta(M_0)$ and is hence not null or co-null. \clubsuit

Lemma 4.4. *Let C be a countable set with a Γ action and let \mathcal{B} equal to either to a Poisson boundary B of Γ or to $B \times B$. If $\psi : \mathcal{B} \rightarrow C$ is a Γ -equivariant measurable map, then ψ is essentially constant.*

Proof. We prove the assertion for $\mathcal{B} = B \times B$. The assertion for $\mathcal{B} = B$ follows then from the first one applied to the precomposition with the projection $\pi_1 : B \times B \rightarrow B$ on the first component.

As the action of Γ in $B \times B$ is ergodic, so is the push-forward measure $\psi_*(\beta \times \beta)$ and hence the image of ψ is supported on an orbit. We now assume that the Γ -action on C is transitive.

If C is finite then there is a finite index subgroup Γ_0 which acts trivially on C . But as the Γ_0 action on B is still ergodic, by Lemma 4.3 we conclude that the action of Γ_0 on C is still transitive and hence C is a single point.

Next, assume that C is infinite. This means that the corresponding generalized Bernoulli action of Γ on 2^C is ergodic (indeed, it is weakly mixing) and measure preserving with respect to the standard Lebesgue measure λ . By the double ergodicity with coefficients of B , [BM02] (see also [BFS, Lemma 2.2]) we conclude that the diagonal Γ -action on $B \times B \times 2^C$ is ergodic. Let $(x, y) \in B \times B$ and $S \subset C$. It is clear that the following evaluation function is essentially constant as it is invariant under the diagonal Γ -action

$$(x, y, S) \mapsto \mathbf{1}_S(\psi(x, y)) \in \{0, 1\}.$$

By Fubini's Theorem, there is a point $(x_0, y_0) \in B \times B$ so that for λ -a.e. $\mathbb{1}_S \in 2^C$ the value of $\mathbb{1}_S(\psi(x_0, y_0))$ is identically 0, or 1. This gives a contradiction. Indeed, for any $c \in C$ we know that

$$\lambda(\{\mathbb{1}_S \in 2^C : \mathbb{1}_S(c) = 0\}) = \lambda(\{\mathbb{1}_S \in 2^C : \mathbb{1}_S(c) = 1\}) = 1/2,$$

in particular for $c_0 := \psi(x_0, y_0)$. ✱

We apply the previous lemma to the countable set $2_f^{\mathfrak{H}(X)}$ consisting of finite subsets of $\mathfrak{H}(X)$.

Corollary 4.5. *Let \mathcal{P} be equal to either $\mathcal{P}(\overline{X})$ or $\mathcal{P}(\overline{X}) \times \mathcal{P}(\overline{X})$. If there exists a Γ -equivariant measurable map $\mathcal{P} \rightarrow 2_f^{\mathfrak{H}(X)}$, then the Γ -action on X is not essential.*

Proof. By hypothesis there is a finite Γ -invariant subset of $\mathfrak{H}(X)$ and in particular, there is a finite Γ -orbit $\Gamma \cdot h$. Then, the corresponding CAT(0) cube complex $X(\Gamma \cdot h)$ is finite and by [CS11, Proposition 3.2, (i) \Rightarrow (iii)], the action is inessential. ✱

4.2. Heavy and Balanced Half Spaces, and Properties of Their Associated Complexes. Let $\mathcal{P}(\overline{X})$ denote the space of probability measures on \overline{X} . If $\mu \in \mathcal{P}(\overline{X})$ define

$$\begin{aligned} H_\mu &:= \{h \in \mathfrak{H}(X) : \mu(h) = \mu(h^*)\} \\ H_\mu^+ &:= \{h \in \mathfrak{H}(X) : \mu(h) > 1/2\} \\ H_\mu^\pm &:= \{h \in \mathfrak{H}(X) : \mu(h) \neq 1/2\}. \end{aligned}$$

We refer to H_μ as to the *balanced* half spaces and to H_μ^+ to the *heavy* half spaces. The terms *unbalanced* and *light* half spaces are also self-explanatory.

We record a few easy consequences of the definition.

Lemma 4.6. *Let $\mu, \nu \in \mathcal{P}(\overline{X})$ be any two measures.*

- (1) *The involution $h \mapsto h^*$ is a bijection between H_μ^+ and H_μ^- .*
- (2) *There is the following partition of half spaces: $\mathfrak{H}(X) = H_\mu \sqcup H_\mu^\pm$, where $H_\mu^\pm = H_\mu^+ \sqcup H_\mu^-$.*
- (3) *If h, k belong to any of the families H_μ , H_μ^+ and H_μ^- , then either $h \triangleleft k$ or h, k are tightly nested.*
- (4) *If $H_\mu \cap H_\nu = \emptyset$, then if $\epsilon \in \{+, -\}$, $H_\mu \cap H_\nu^\epsilon \neq \emptyset$.*
- (5) *There are no facing triple of half spaces in H_μ . If X is not Euclidean it follows that $H_\mu^+ \neq \emptyset$.*
- (6) *If $h, k \in H_\mu$ are two parallel half spaces with $h \subset k$ then $\mu(h^* \cap k) = 0$.*
- (7) *The assignments $\mu \mapsto H_\mu$ and $\mu \mapsto H_\mu^\epsilon$, for $\epsilon \in \{+, -\}$, are $\text{Aut}(X)$ -equivariant for the natural actions on $\mathcal{P}(\overline{X})$ and $2^{\mathfrak{H}(X)}$.*

Proof. Assertions (1), (2) and (3) are obvious. Assertion (4) follows from the fact that if $H_\mu \cap H_\nu = \emptyset$, then $H_\mu \subset H_\mu^\pm$. But then, since H_μ is invariant under the involution $h \mapsto h^*$, both $H_\mu \cap H_\nu^+$ and $H_\mu \cap H_\nu^-$ must be non-empty.

To see (5), assume that h_1, h_2, h_3 were a facing triples of half-spaces in H_μ , so that $h_2^* \subset h_1$, $h_3^* \subset h_1$ and $\hat{h}_2 \parallel \hat{h}_3$. This would imply that $1/2 = \mu(h_1) \geq \mu(h_2^*) + \mu(h_3^*) = 1$, a contradiction. Since X is not Euclidean, and hence there are facing triple of half-spaces, then $H_\mu^\pm \neq \emptyset$ and also $H_\mu^+ \neq \emptyset$.

Assertion (6) is immediate since $\mu(k) = \mu(h^*) + \mu(h \cap k)$ and $h^*, k \in H_\mu$ and (7) is immediate from the definitions. \clubsuit

Definition 4.7. If $\pi : \mathfrak{H}(X) \rightarrow \hat{\mathfrak{H}}(X)$ is the projection, let $\hat{H}_\mu := \pi(H_\mu)$ and $\hat{H}_\mu^\pm := \pi(H_\mu^\pm)$ be respectively the *balanced* and the *unbalanced* hyperplanes. Define

$$\overline{X}_\mu := \left\{ \alpha : \hat{\mathfrak{H}}(X) \rightarrow \mathfrak{H}(X) : \mu(\alpha(\hat{h})) > \frac{1}{2} \text{ for all } \hat{h} \in \hat{H}_\mu^\pm \text{ and } \alpha \text{ is an admissible section} \right\}$$

and recall that the complex $\overline{X}(H_\mu)$ is defined as

$$\overline{X}(H_\mu) := \{ \alpha_0 : \hat{H}_\mu \rightarrow H_\mu : \alpha_0 \text{ is an admissible section} \}.$$

Lemma 4.8. *For every $\mu \in \mathcal{P}(\overline{X})$ the subcomplex \overline{X}_μ is not empty and it is isometrically isomorphic to the abstract complex $\overline{X}(H_\mu)$. Moreover if $H_\mu = \emptyset$, then \overline{X}_μ is a single vertex in \overline{X} .*

Proof. Just by chasing the definitions one can easily see that

$$(4.17) \quad \overline{X}_\mu = \bigcap_{h \in H_\mu^+} h.$$

In fact,

$$\begin{aligned} & \alpha \in \overline{X}_\mu \\ \iff & \alpha : \hat{\mathfrak{H}}(X) \rightarrow \mathfrak{H}(X) \text{ is an admissible section such that } \mu(\alpha(\hat{h})) > \frac{1}{2} \text{ for all } \hat{h} \in \hat{H}_\mu^\pm \\ \iff & \alpha(\hat{h}) \in H_\mu^+ \text{ for all } \hat{h} \in \hat{H}_\mu^\pm \\ \iff & \alpha \in \bigcap_{h \in H_\mu^+} h. \end{aligned}$$

The fact that $\overline{X}_\mu \neq \emptyset$ follows from Lemma 4.6(5). The restriction $\alpha \mapsto \alpha|_{\hat{H}_\mu}$ is a map

$$\overline{X}_\mu \rightarrow \overline{X}(H_\mu), \text{ whose inverse is defined by } \alpha_0 \mapsto \alpha = \begin{cases} \alpha_0 & \text{on } \hat{H}_\mu^\pm \\ h \in H_\mu^+ & \text{for } \hat{h} \in \hat{H}_\mu^\pm. \end{cases}$$

If $H_\mu = \emptyset$, then $\hat{H}_\mu^\pm = \hat{\mathfrak{H}}(X)$, and hence \overline{X}_μ , consist of the unique admissible section α that chooses the heavy half space for any hyperplane. \clubsuit

Lemma 4.9. *The complex $\overline{X}(H_\mu)$ is an interval.*

Proof. Let us consider the projection $p : \overline{X} \rightarrow \overline{X}(H_\mu)$ and let $\alpha_0 \in \text{supp}(p_*\mu)$. Let α_0^* be the “opposite” of α_0 (in H_μ). Observe that α_0^* is an ultrafilter on H_μ : indeed, the only nontrivial condition we must check is that if $h \in \alpha_0^*$ and $h \subset k$ then $k \in \alpha_0^*$. If instead $k \notin \alpha_0^*$, then $k \in \alpha_0$ which means that $h^* \cap k$ is an open neighborhood of α_0 , contradicting that α_0 is in the support of μ with Lemma 4.6(6). By construction, it is clear that $H_\mu = [\alpha_0, \alpha_0^*] \cup [\alpha_0^*, \alpha_0]$, where the intervals are taken in $\overline{X}(H_\mu)$. \clubsuit

Definition 4.10. Let \mathfrak{H}' be a subset of $\mathfrak{H}(X)$. An element $h \in \mathfrak{H}'$ is called:

- *minimal in \mathfrak{H}'* if for every $k \in \mathfrak{H}'$ either $k \supset h$, $h \subset k$, or $h \subset k^*$;
- *maximal in \mathfrak{H}'* if for every $k \in \mathfrak{H}'$ either $k \supset h$, $k \subset h$, or $k^* \subset h$, that is to say, h is maximal if h^* is minimal;
- *terminal in \mathfrak{H}'* if it is either maximal or minimal.

Remark 4.11. The number of terminal elements is bounded above by $2d$ not just for H_μ but for any union of tightly nested pairwise incomparable chains in H_μ .

4.3. Proof of Theorem 4.1.

Proof of Theorem 4.1. Since the Γ -action on (B, ϑ) is amenable, there exists a Γ -equivariant measurable map $\psi : B \rightarrow \mathcal{P}(\overline{X})$ into the probability measures on \overline{X} . We consider $\mathcal{P}(\overline{X})$ endowed with the push-forward of the quasi-invariant, doubly-ergodic measure ϑ on B , so that Γ acts ergodically on $\mathcal{P}(\overline{X}) \times \mathcal{P}(\overline{X})$. We will show that under the hypotheses of the theorem, we may associate to every μ in the image of ψ a point in ∂X and the composition will be the required Γ -equivariant measurable boundary map $\varphi : B \rightarrow \partial X$.

The map $C_1 : \mathcal{P}(\overline{X}) \rightarrow \mathbb{N} \cup \{\infty\}$ defined by $\mu \mapsto |H_\mu|$ is measurable (Corollary A.2(1)) and Γ -equivariant, hence by ergodicity it is essentially constant.

I. $H_\mu = \emptyset$ for almost all μ

If the essential value of C_1 is 0, then for almost every $\mu \in \mathcal{P}(\overline{X})$, $H_\mu = \emptyset$. This means that, up to measure 0, the image of ψ lies in the set $\mathcal{E} := \{\mu \in \mathcal{P}(\overline{X}) : H_\mu = \emptyset\}$. Thus we have a well defined composition $\varphi : B \rightarrow \mathcal{E} \rightarrow \overline{X}$ defined by $x \mapsto \psi(x) \mapsto \overline{X}_{\psi(x)}$, whose image is the single point $\overline{X}_{\psi(x)} \in \overline{X}$ (Lemma 4.8). Measurability is guaranteed by Lemma A.1, and Lemma A.3. The equivariance under Γ follows from Lemma 4.6(7). Proposition 4.13 will show that, in fact, φ takes values into the non-terminating ultrafilters of X .

The rest of the proof will consist in showing that all other cases cannot occur.

II. $0 < |H_\mu| < \infty$ for almost all μ

If the essential value of C_1 were to be finite, then Corollary 4.5 with $\mathcal{P} = \mathcal{P}(\overline{X})$ would imply that the action is not essential.

III. $|H_\mu| = \infty$ for almost all μ

To deal with this case we consider the Γ -equivariant and measurable function $C_2 : \mathcal{P}(\overline{X}) \times \mathcal{P}(\overline{X}) \rightarrow \mathbb{N} \cup \{\infty\}$, defined by $(\mu, \nu) \mapsto |H_\mu \cap H_\nu|$ (Corollary A.2((2)). Again by ergodicity of the Γ -action on $\mathcal{P}(\overline{X}) \times \mathcal{P}(\overline{X})$, the function C_2 is essentially constant.

III.a $0 < |H_\mu \cap H_\nu| < \infty$ for almost all μ, ν

If the essential value of C_2 were finite and non-zero, then Corollary 4.5 with $\mathcal{P} = \mathcal{P}(\overline{X}) \times \mathcal{P}(\overline{X})$ would again imply that the action is not essential.

III.b $|H_\mu \cap H_\nu| = 0$ for almost all μ, ν

Now suppose that the essential value of C_2 is 0, so that for almost every $\mu, \nu \in \mathcal{P}(\overline{X})$, $H_\mu \cap H_\nu = \emptyset$. Let us consider the measurable (Corollary A.2(3)) Γ -equivariant function $T : \mathcal{P}(\overline{X}) \times \mathcal{P}(\overline{X}) \rightarrow \mathbb{N} \cup \{\infty\}$, defined by

$$T(\mu, \nu) := |\tau[(H_\mu \cap H_\nu^+) \cup (H_\nu \cap H_\mu^+)]|,$$

where

$$(4.18) \quad \tau : 2^{\mathfrak{H}(X)} \rightarrow 2^{\mathfrak{H}(X)}$$

is the map that assigns to a subset of half spaces its terminal elements. By double ergodicity T is essentially constant. Using the fact that both H_μ and H_ν are Euclidean, any subset of them must have finitely many terminal elements and therefore this essential value must be finite (see Remark 4.11). Once more, essentiality of the action, along with Corollary 4.5 assures us that the essential value is 0.

This leaves us with the case in which the essential value is zero, that is $H_\mu \cap H_\nu^+$ has no terminal elements for almost every (μ, ν) . In this case the following proposition (whose proof we postpone to § 4.4) allows us to conclude that this case cannot happen.

Proposition 4.12. *Suppose that for almost every $\mu, \nu \in \mathcal{P}(\overline{X})$, $|H_\mu| = |H_\nu| = \infty$, $H_\mu \cap H_\nu = \emptyset$ and $H_\mu \cap H_\nu^+$ has no minimal elements. Then X contains cubes of arbitrarily large dimension.*

III.c $|H_\mu \cap H_\nu| = \infty$ for almost all μ, ν

Finally, let us suppose that the essential value of C_2 is ∞ , namely $|H_\mu \cap H_\nu| = \infty$ for almost every $(\mu, \nu) \in \mathcal{P}(\overline{X}) \times \mathcal{P}(\overline{X})$.

If $H_\mu = H_\nu$ for almost every $\mu, \nu \in \mathcal{P}(\overline{X})$, then applying Fubini, there is a $\mu_0 \in \mathcal{P}(\overline{X})$ such that for every ν in a co-null Γ -invariant subset we have that $H_{\mu_0} = H_\nu$. Hence, $H_{\gamma_*\nu} = \gamma H_\nu = H_\nu$. Since the action is essential without fixed points on the visual boundary, we may flip any $h \in H_\nu^+$. This means that $H_\nu^- \cap H_{\gamma_*\nu}^+ \neq \emptyset$ and so H_ν^+ is not Γ -invariant. As a result the corresponding embedded subcomplexes $\overline{X}_{\gamma_*\nu}$ are not invariant. We will see in Proposition 4.19 that this implies that X is a product, which is a contradiction.

We are therefore left in the case in which $H_\mu \cap H_\nu$ is infinite but $H_\mu \neq H_\nu$, for almost all $\mu, \nu \in \mathcal{P}(\overline{X})$.

We now consider whether or not H_μ has strongly separated half spaces. Observe that the set

$$\mathcal{S} = \{(h_1, h_2) \in \mathfrak{H}(X) \times \mathfrak{H}(X) : h_1, h_2 \text{ are strongly separated}\}$$

is Γ -invariant. Therefore, the map $\mu \rightarrow |(H_\mu \times H_\mu) \cap \mathcal{S}|$ is measurable (Corollary A.2(4)) and Γ -invariant, and hence essentially constant.

If H_μ contains pairs of strongly separated half spaces then H_μ^+ satisfies the Descending Chain Condition (Lemma 4.20). This implies that the action is again inessential by extracting the finitely many terminal elements of the set $[H_\mu^+ \cap (H_\nu \setminus H_\mu)] \cup [H_\nu^+ \cap (H_\mu \setminus H_\nu)]$ (Corollary A.2(5)), and we proceed as before to conclude that the action is inessential.

If on the other hand H_μ does not contain pairs of strongly separated half spaces, then by Corollary 4.23 the action is inessential. \clubsuit

4.4. Further Properties and Proofs.

Proposition 4.13. *Let X be a finite dimensional $CAT(0)$ cube complex, $\Gamma \rightarrow \text{Aut}(X)$ an essential action on X , (B, ν) a doubly ergodic Γ -space with quasi-invariant measure ν and $\varphi : B \rightarrow \overline{X}$ a measurable Γ -equivariant map. Then φ takes values in the non-terminal ultrafilters of X .*

We start the proof with few easy observations. If $\alpha, \beta \in \mathcal{U}$, recall that

$$\mathfrak{H}(\alpha, \beta) := [\alpha, \beta] \cup [\beta, \alpha] = [\alpha, \beta] \cup [\alpha, \beta]^*.$$

Then it is easy to check that

$$(4.19) \quad \tau(\mathfrak{H}(\alpha, \beta)) = \tau([\alpha, \beta]) \cup \tau([\alpha, \beta]^*)$$

and hence $|\tau(\mathfrak{H}(\alpha, \beta))|$ is finite.

Moreover, we can make the following easy observation:

Lemma 4.14. *Let $\alpha, \beta \in \mathcal{U}$ and $h \in \tau(\alpha)$. Then $\beta \notin h$ if and only if $h \in \tau(\mathfrak{H}(\alpha, \beta))$.*

Proof. If $\beta \in h$, then h does not separate α and β , so that $h \notin \mathfrak{H}(\alpha, \beta)$ and, even more so, $h \notin \tau(\mathfrak{H}(\alpha, \beta))$. The converse is equally easy and will not be needed. \clubsuit

Proof. We may assume that X is irreducible. The general case will follow from this case as in the proof of Corollary 4.2, since the set of non-terminating ultrafilters in a product is the cartesian product of the sets of non-terminating ultrafilters of each factor.

The composition of φ with the map τ defined in (4.18) that assigns to a set of half-spaces its terminal element, gives a Γ -equivariant measurable map $B \rightarrow 2^{\mathfrak{H}(X)}$ defined by $x \mapsto \tau(\phi(x))$. The function $C_4 : B \rightarrow \mathbb{N} \cup \{\infty\}$ defined by $x \mapsto |\tau(\phi(x))|$ is hence essentially constant.

Therefore, we want to show that $|C_4(x)| = 0$ for almost every x , that is that the set $\tau(\phi(x))$ is empty, thus showing that $\varphi(x)$ is non-terminating.

To this purpose let us consider the map $\theta : B \times B \rightarrow 2^{\mathfrak{H}(X)}$ that to a pair $(x, y) \in B \times B$ associates the set of terminal elements in $H(\varphi(x), \varphi(y))$. Again by ergodicity the function $C_5 : B \times B \rightarrow \mathbb{N} \cup \{\infty\}$, defined by $C_5(x, y) := |\tau(H(\varphi(x), \varphi(y)))|$ is essentially constant and, by the observation right after (4.19), $0 \leq |\tau(H(\varphi(x), \varphi(y)))| < \infty$.

By Corollary 4.5 with $\mathcal{P} = B$, we deduce that for almost every $x, y \in B$, $\tau(H(\varphi(x), \varphi(y))) = \emptyset$. We show now that this is incompatible with $|\tau(\varphi(x))| > 0$ for almost every $x \in B$, thus proving the proposition.

Let $x_0 \in B$ be such that $|\tau(\varphi(x_0))| > 0$ and let $B_0 \subset B$ be a set of full measure such that $\tau(H(\varphi(x_0), \varphi(y))) = \emptyset$ for all $y \in B_0$. Then by Lemma 4.14, if $\tau(\varphi(x_0)) \in h$, we must have that $\varphi(y) \in h$ for all $y \in B_0$. But B_0 contains a Γ -orbit and hence this contradicts the fact that the action is essential. \clubsuit

4.4.1. Proof of Proposition 4.12 (in step III.b). We will find arbitrarily a large family of pairwise intersecting half-spaces. To this purpose, choose a sequence $\{\mu_i\}_{i \in \mathbb{N}}$ of pairwise generic measures satisfying the hypotheses of Proposition 4.12. For each i , choose an infinite descending chain $h_n^i \in H_{\mu_0}^+ \cap H_{\mu_i}$.

Consider the following property of an ordered pair (μ_i, μ_j) of measures:

- (*) There exists $C(i, j) \in \mathbb{N}$ such that for every $n \geq C(i, j)$ there is an $M_n \geq C(i, j)$ such that if $m > M_n \geq C(i, j)$, then $\hat{h}_n^i \cap \hat{h}_m^j$.

Lemma 4.15. *Up to switching i and j , any pair of measures μ_i and μ_j , satisfies (*).*

We postpone the proof of this lemma and show how to conclude the proof.

Let us consider a graph $\mathcal{G} := \mathcal{G}(V, E)$, where $V := \{\mu_i\}$ and where two measures μ_i and μ_j are connected by an edge $e \in E$ with source μ_i and target μ_j if the ordered pair (μ_i, μ_j) satisfies (*). By Lemma A.8, given $D \in \mathbb{N}$, there exist (relabelled) measures $\mu_1, \dots, \mu_D \in \{\mu_i\}_{i \in \mathbb{N}}$ such that for $1 \leq i < j \leq D$, each ordered pair (μ_i, μ_j) satisfies (*).

By choosing

$$C := \max\{C(i, j) : 1 \leq i < j \leq D\}$$

and

$$M := \max\{M_C(i, j) : 1 \leq i < j \leq D\}.$$

we obtain that for all $n, m \geq C$ and $1 \leq i, j \leq D$, the corresponding hyperplanes are transverse, $\hat{h}_n^i \cap \hat{h}_m^j$. The next proof will hence conclude the proof of Proposition 4.12.

Proof of Lemma 4.15. Fix two measure that we denote for ease of notation, μ and ν . Let $h_n \in H_{\mu_0}^+ \cap H_\mu$ and $k_m \in H_{\mu_0}^+ \cap H_\nu$ be the corresponding infinite descending sequences. Since all the half spaces in question belong to $H_{\mu_0}^+$, for each pair n, m we have the following decomposition

$$\mathbb{N} \times \mathbb{N} = N_1 \sqcup N_2 \sqcup N_3 \sqcup N_4,$$

where

$$\begin{aligned} N_1 &= \{(n, m) : h_n \cap k_m\} \\ N_2 &= \{(n, m) : h_n^* \subset k_m\} \\ N_3 &= \{(n, m) : h_n \subset k_m\} \\ N_4 &= \{(n, m) : h_n \supset k_m\} \end{aligned}$$

We claim that if we allow ourselves to throw away a finite number of pairs (n, m) if necessary, then the decomposition of $\mathbb{N} \times \mathbb{N}$ takes in fact a simpler shape. Namely:

Claim 4.16. *There exists a constant $C \in \mathbb{N}$ depending on μ and ν , such that*

$$\mathbb{N}_C := (\mathbb{N} \times \mathbb{N}) \cap ([C, \infty) \times [C, \infty)) = N_1 \sqcup N_j,$$

where $j = 2, 3$ or 4 .

In fact, let us suppose that $N_2 \neq \emptyset$ and $N_3 \neq \emptyset$ and let us take $(n_3, m_3) \in N_3$ and $(n, m) \in N_2$. Set $m' := \min\{m, m_3\}$, such that

$$h_n^* \subset k_{m'} \quad \text{and} \quad h_{n_3} \subset k_{m'}.$$

If $n \geq n_3$, then $h_n \subset h_{n_3} \subset k_{m'}$, which is impossible since also $h_n^* \subset k_{m'}$. Hence there is no pair $(n, m) \in N_2$ such that $n \geq \min\{n_3 : (n_3, m_3) \in N_3\} =: A_3$. It follows that

$$(4.20) \quad \{(n, m) \in N_2 : n \geq A_3\} \cap N_3 = \emptyset.$$

Now let us suppose that $N_3 \neq \emptyset$ and $N_4 \neq \emptyset$ and let us take $(n_3, m_3) \in N_3$ and $(n, m) \in N_4$. If $n \geq n_3$, then

$$k_n \subset h_n \subset h_{n_3} \subset k_{m_3},$$

which is impossible since the sequences of half-spaces are tightly nested. Hence, analogously to the previous case, we have that

$$(4.21) \quad \{(n, m) \in N_2 : n \geq A_3\} \cap N_4 = \emptyset.$$

Finally, let us suppose that $N_2 \neq \emptyset$ and $N_4 \neq \emptyset$ and let us take $(n, m) \in N_2$ and $(n_4, m_4) \in N_4$. Set $n' := \min\{n, n_4\}$, so that

$$h_{n'} \supset k_m \quad \text{and} \quad h_{n'} \supset k_{m_4}.$$

If $m \geq m_4$ then $h_{n'} \supset k_{m_4} \subset k_m$, which is impossible since also $h_{n'} \supset k_m^*$. Hence there is no pair $(n, m) \in N_2$ such that $m < \min\{m_4 : (n_4, m_4) \in N_4\} =: B_4$. It follows that

$$(4.22) \quad \{(n, m) \in N_2 : m \geq B_4\} \cap N_4 = \emptyset.$$

By setting $C := \max\{A_3, B_4\}$ we have proven the claim.

Let us suppose now that for $n_0, m_0 \geq C$, the pair $(n_0, m_0) \in N_1 \sqcup N_3$ and, in fact, that $(n_0, m_0) \in N_3$ (otherwise there is nothing to prove). Choose $m_0 = m_0(n_0)$ to be the largest integer such that $(n_0, m_0) \in N_3$. Then, because $\mathbb{N}_C = N_1 \sqcup N_3$, for every $m \geq m_0(n_0) + 1$, the hyperplanes \hat{h}_{n_0} and \hat{k}_m are transverse. Hence the assertion of the lemma is proven in the case in which $\mathbb{N}_C = N_1 \sqcup N_3$.

Remark that the same identical argument shows the assertion if $\mathbb{N}_C = N_1 \sqcup N_2$, since we only used that there is a minimal element k_{m_0} in the sequence k_m that contains the hyperplane \hat{h}_{n_0} .

The argument if $\mathbb{N}_C = N_1 \sqcup N_4$ is analogous, but with the role of h_n and k_m reversed, as now there is a minimal element h_{n_0} in the sequence h_n that contains the hyperplane \hat{k}_{m_0} . Namely, let $(n_0, m_0) \in N_4$ be such a pair. Then for all $n \geq n_0(m_0) + 1$, the hyperplanes \hat{h}_n and \hat{k}_{m_0} are transverse. \clubsuit

Remark 4.17. The last assertion in the proof relative to the case $\mathbb{N}_C = N_1 \sqcup N_4$ holds also for the case $\mathbb{N}_C = N_1 \sqcup N_2$, but the symmetry of this case is not useful.

4.4.2. *Proofs needed in step III.c.* Recall that a set Y is called *strongly convex* if for any $x, y \in Y$ then $[x, y] \subset Y$.

Lemma 4.18. *Let X be a $CAT(0)$ cube complex and A be any cubical subset of X (that is, A is a union of cubes, not necessarily connected). If Y be the smallest strongly convex subcomplex of X containing A , then*

$$\hat{\mathfrak{H}}(Y) = \hat{\mathfrak{H}}(A) \sqcup \{\hat{h} \in \hat{\mathfrak{H}}(Y) : \hat{h} \text{ separates } A \text{ in at least two non-trivial subsets}\}.$$

Proof. We only need to check that every $\hat{h} \in \hat{\mathfrak{H}}(Y) \setminus \hat{\mathfrak{H}}(A)$ separates A in non-trivial subsets. Take $\hat{h} = (h, h^*) \in \hat{\mathfrak{H}}(Y)$ and assume by contradiction that $A \subseteq h$. Then any geodesic between two points of A is also contained in h (otherwise this geodesic would cross h twice). Hence $Y \subseteq h$, contradicting that $h \in \hat{\mathfrak{H}}(Y)$. \clubsuit

Proposition 4.19. *Let X be a $CAT(0)$ cube complex and $\Gamma \rightarrow \text{Aut}(X)$ an essential action. Let $\hat{\mathfrak{H}}' \subset \hat{\mathfrak{H}}(X)$ be a Γ -invariant subset of half-spaces and $X_\alpha \subset X$ a Γ -invariant family of subcomplexes such that $\hat{\mathfrak{H}}(X_\alpha) = \hat{\mathfrak{H}}'$. Let Y be the smallest strongly convex subcomplex containing $A := \cup_\alpha X_\alpha$. Then $Y = X$ and $\overline{X} = \overline{X(\hat{\mathfrak{H}}')} \times Z$.*

Proof. Since Y is Γ -invariant and the action is essential, then $Y = X$.

Because of Lemma 4.18, the hyperplanes of Y are of two types: either they are in $\hat{\mathfrak{H}}' = \hat{\mathfrak{H}}(A)$ and they separate one (equivalently, any) of the X_α or they separate a X_α from a $X_{\alpha'}$. Any hyperplane \hat{h} of this second type will cross any hyperplane $\hat{k} \in \hat{\mathfrak{H}}'$. Indeed, if $\hat{h} = (h, h^*)$ and $\hat{k} = (k, k^*)$ it is easy to see that the four intersections in (2.2) are non-empty. Hence X is a product. \clubsuit

Lemma 4.20. *If $|H_\mu| = \infty$ and H_μ contains strongly separated half spaces, then H_μ^+ satisfies the Descending Chain Condition.*

Proof. Let $h, k \in \mathfrak{H}(X)$ be a pair of strongly separated half spaces in H_μ with $h \subset k$. There is the following decomposition

$$(4.23) \quad H_\mu^+ = P(h) \cup P(k),$$

where $P(h)$ and $P(k)$ are the μ -heavy half spaces that are parallel respectively to h and k . Notice that, while $P(h)$ and $P(k)$ are not necessarily disjoint, their union is the whole of H_μ^+ since h and k are strongly separated.

Let $h_n \in H_\mu^+$ be a descending chain, i.e. $h_{n+1} \subset h_n$. We must show that the chain terminates. By passing to a subsequence, we may assume that h_n belong to the same set for all $n \in \mathbb{N}$ and it is hence enough to consider for example the case $h_n \in P(h)$ for all $n \in \mathbb{N}$.

Since $h_n \in H_\mu^+$ and $h \in H_\mu$, we cannot have that $h_n \subset h$ or $h \subset h_n^*$. Let us suppose that $h \subset h_n$. Since between h and h_n there are only finitely many half-spaces, and since no μ -heavy half space can be contained in a balanced one, the chain must terminate. Likewise the same argument applied to $h^* \subset h_n$ shows that the chain must terminate. \ast

Lemma 4.21. *For every measure μ either \hat{H}_μ contains a pair of strongly separated hyperplanes or there exists a pair $h \in H_\mu^-, k \in H_\mu^+$ of half spaces, such that the hyperplanes \hat{h} and \hat{k} are strongly separated and for every $x \in H_\mu$, $\hat{x} \subset h^* \cap k$.*

Proof. Suppose that H_μ does not contain strongly separated half spaces. We first show that for every $x \in H_\mu$, there exist $k_0(x), k_3(x) \in H_\mu^\pm$ such that $\hat{k}_0(x)$ and $\hat{k}_3(x)$ are strongly separated and $\hat{x} \subset k_0^*(x) \cap k_3(x)$. For ease of notation we drop the dependence on x .

In fact, since X is irreducible, given $x \in H_\mu$, there exist half spaces k_1, k_2 such that \hat{k}_1 and \hat{k}_2 are strongly separated hyperplanes and $k_1 \subset x \subset k_2$. Then at least one between the k_1 and k_2 must be in H_μ^\pm , but perhaps not both of them. Then double skewer k_2 into k_1 and k_1^* into k_2^* to obtain

$$\gamma k_2 \subset k_1 \subset x \subset k_2 \subset \gamma^{-1} k_1,$$

where the pairs $\gamma k_2, k_2$ and $k_1, \gamma^{-1} k_1$ are strongly separated. Since all hyperplanes corresponding to pairs of half spaces in the sequence $k_0 \subset k_1 \subset k_2 \subset k_3$ are strongly separated, there can be at most one half space that belongs to H_μ . By measure considerations, this half space can only be either k_1 or k_2 , so that $k_0, k_3 \in H_\mu^\pm$, and the assertion is proven. In particular $\gamma k_1 \in H_\mu^-$ and $\gamma^{-1} k_2 \in H_\mu^+$.

Double skewer once again to get $h \in H_\mu^-$ and $k \in H_\mu^+$ with \hat{h}, \hat{k} strongly separated, such that

$$h \subset k_0 \subset x \subset k_3 \subset k.$$

We show now that, given any $y \in H_\mu$, we have $\hat{y} \subset h^* \cap k$. In fact, we cannot have $k \subset y$ or $k \subset y^*$, since $y, y^* \in H_\mu$ and $k \in H_\mu^+$. Analogously, we cannot have that $\hat{y} \supset \hat{k}$, because otherwise \hat{y} could not intersect \hat{k}_3 and hence it would have to contain it, which is impossible again by measure considerations. Hence $y \subset k$. An analogous argument shows that $h \subset y$, thus completing the proof. \ast

The above argument can be extended to show the following:

Lemma 4.22. *Let $\mu_i \in \mathcal{P}(\overline{X})$ be measures such that \hat{H}_{μ_i} does not contain strongly separated hyperplanes for all i and $H_{\mu_i} \cap H_{\mu_j} \neq \emptyset$ for all i, j . Then there exists a pair of half spaces $h \subset k$ such that \hat{h}, \hat{k} are strongly separated and, for every $x \in H_{\mu_j}$, $\hat{x} \subset h^* \cap k$.*

Proof. Fix μ_0 and apply Lemma 4.21 to find half spaces $h_2 \subset h_3$ such that \hat{h}_2, \hat{h}_3 are strongly separated and

$$(4.24) \quad \hat{x} \subset h_2^* \cap h_3.$$

Use the double skewering lemma several times to find a chain $h_0 \subset \dots \subset h_5$ of half spaces with corresponding pairwise strongly separated hyperplanes. We will use that (4.24) holds in particular for every $x_j \in H_{\mu_0} \cap H_{\mu_i}$ to show that $\hat{y} \subset h_0^* \cap h_5$ for every $y \in H_{\mu_j}$ and every j .

Consider in fact $y \in H_{\mu_i}$. Observe that \hat{y} can be transverse to at most one \hat{h}_i , $0 \leq i \leq 5$, since these are pairwise strongly separated. If it is transverse to any \hat{h}_i for $1 \leq i \leq 4$, we are done, since then $\hat{y} \subset h_0^* \cap h_5$. Suppose instead that \hat{y} is transverse to \hat{h}_0 . Then \hat{h}_1 and h_2 are nested in between \hat{y} and \hat{x}_j , which is impossible because the hyperplanes in \hat{H}_{μ_j} are tightly nested and \hat{H}_{μ_j} does not contain strongly separated hyperplanes. A similar argument shows that \hat{y} cannot be transverse to \hat{h}_5 .

If instead \hat{y} is parallel to all \hat{h}_i , for $0 \leq i \leq 5$, then we have to check that $\hat{y} \subset h_0$ and $\hat{y} \subset h_5^*$ cannot happen. In fact, if $\hat{y} \subset h_0$, as before this would force \hat{h}_1, \hat{h}_2 to be in \hat{H}_{μ_j} , which is not possible because they are a strongly separated pair. The case in which $\hat{y} \subset h_5^*$ can be excluded analogously. \spadesuit

Corollary 4.23. *Assume that for almost every μ , there are no strongly separated pairs in H_μ . If $H_\mu \cap H_\nu \neq \emptyset$ for almost every pair (μ, ν) then the Γ -action is non-essential.*

Proof. Fix a generic measure μ_0 with a generic Γ -invariant set B_0 such that for every $\nu \in B_0$ we have that $H_{\mu_0} \cap H_\nu \neq \emptyset$.

Lemma 4.22 implies the existence of a pair of half spaces $h \subset k$ such that $h^* \cap k$ contains all the hyperplanes in \hat{H}_μ for $\nu \in B_0$, in particular those in $\gamma \hat{H}_{\mu_0} = \mathfrak{H}(X)_{\gamma * \mu_0, b}$ for all $\gamma \in \Gamma$. This shows that the two half spaces h and k are not Γ -flippable, which contradicts either that the action is essential or that it is without fixed points on the CAT(0) boundary [CS11, Theorem 4.1]. \spadesuit

5. PROOF OF THEOREM 1.1

Let X be a finite dimensional CAT(0) cube complex, $\Gamma \rightarrow \text{Aut}(X)$ a non-elementary action and $Y \subset X$ its essential core. Let (B, ϑ) be any Poisson boundary of Γ . In order to prove our main result, we constructed in § 4 a measurable Γ -equivariant boundary map $\varphi : B \rightarrow \partial X$

to the Roller boundary ∂X . The precomposition of the median cocycle c with $\varphi : B \rightarrow \partial X$ yields a Γ -equivariant cocycle defined on B^3 , which we will show is non-zero on a set of positive measure (Proposition 5.1 and Lemma 5.3). According to (3.15), this ensures the existence of a non-trivial cohomology class on Γ . Then [BI02] ensures that the median class of the Γ -action $\rho^*(\mathfrak{m}_n) \in H_b^2(\Gamma, \ell^p(\mathfrak{H}(X)^{(n)}))$ corresponds to the cohomology class $c \circ \varphi^3$ on B^3 and hence does not vanish.

5.1. Passing from a Cocycle on ∂X to a Cocycle on B . We give a condition on a Γ -equivariant cocycle $d : (\partial X)^3 \rightarrow E$ to guarantee that $d \circ \varphi^3 : B^3 \rightarrow E$ is non-zero on a set of positive measure.

Let K be compact metrizable Γ -space. A measure $\lambda \in \mathcal{P}(K)$ is quasi-invariant if λ and $\gamma_*\lambda$ have the same null sets, for all $\gamma \in \Gamma$.

If $h \in \mathfrak{H}(X)$ is a half space, we set

$$\bar{h} := \{x \in \bar{X} : x \in h\}$$

and

$$\partial h := \bar{h} \cap \partial X.$$

Proposition 5.1. *Let Γ be a group with a non-elementary and essential action $\Gamma \rightarrow \text{Aut}(Y)$ on a finite dimensional CAT(0) cube complex Y . If (B, ϑ) is a Poisson boundary of Γ , let $\varphi : B \rightarrow \partial Y$ be a Γ -equivariant measurable map. Let $d : (\partial Y)^3 \rightarrow E$ be an everywhere defined alternating bounded Γ -equivariant Borel cocycle with values in a coefficient Γ -module E . If there exist half spaces $h_i \in \mathfrak{H}(Y)$ such that $d(\xi_1, \xi_2, \xi_3) \neq 0$ for every $(\xi_1, \xi_2, \xi_3) \in \partial h_1 \times \partial h_2 \times \partial h_3$ then $d \circ \varphi^3$ is a non-trivial element of $\mathcal{ZL}_{\text{alt},*}^\infty(B^3, E)^\Gamma$.*

The proof is almost an immediate consequence of the following:

Lemma 5.2. *Let Y be a finite dimensional CAT(0) cube complex and $\Gamma \rightarrow \text{Aut}(Y)$ a non-elementary essential action. If $\lambda \in \mathcal{P}(\partial Y)$ is any quasi-invariant probability measure then $\lambda(\partial h) > 0$ for any half space $h \in \mathfrak{H}(Y)$.*

Proof. If $\lambda(\partial h) = 0$ then $\lambda(\partial h^*) = 1$. By the Flipping Lemma [CS11, Theorem 4.1], there exists $\gamma \in \Gamma$ such that $h^* \subset \gamma h$. But this is a contradiction because $\partial h^* \subset \partial(\gamma h) = \gamma \partial h$, while, by quasi-invariance, $\lambda(\gamma \partial h) = 0$. \spadesuit

Proof of Proposition 5.1. If there exist half spaces $h_i \in \mathfrak{H}(Y)$ such that $d(\xi_1, \xi_2, \xi_3) \neq 0$ for every $(\xi_1, \xi_2, \xi_3) \in \partial h_1 \times \partial h_2 \times \partial h_3$ then $d \circ \varphi^3(x_1, x_2, x_3) \neq 0$ for almost every $(x_1, x_2, x_3) \in \varphi^{-1}(\partial h_1) \times \varphi^{-1}(\partial h_2) \times \varphi^{-1}(\partial h_3) =: S \subset B^3$.

By Lemma 5.2 applied to the quasi-invariant probability measure $\varphi_*\vartheta \in \mathcal{P}(\partial Y)$, the set S has positive ϑ^3 -measure and hence $d \circ \varphi^3$ it is a non-trivial element of $\mathcal{ZL}_{\text{alt},*}^\infty(B^3, E)^\Gamma$. \spadesuit

5.2. Proof of Theorem 1.1. Let $n \geq 2$ and denote by $(\partial X)^{[3]}$ is the set of triples of ∂X that can be separated by an n -disjoint facing triple of half spaces,

$$(\partial Y)^{[3]} = \{(\xi_1, \xi_2, \xi_3) \in \partial Y \times \partial Y \times \partial Y : \text{there exist } h_i \in \mathfrak{H}(Y), \text{ such that } \xi_i \in h_i, \\ i = 1, 2, 3, \text{ and the } h_i \text{ are an } n\text{-disjoint facing triple}\}.$$

Recall that, if the action of Γ is non-elementary, then the Γ -essential core $Y \subset X$ is not empty. Then, according to Lemma 2.17, for every $h \in \mathfrak{H}(Y)$ there exist $\gamma, \gamma' \in \Gamma$ such that $h, \gamma h, \gamma' h$ are n -disjoint, so that

$$A_h := \partial h \times \partial(\gamma h) \times \partial(\gamma' h) \subset (\partial Y)^{[3]}.$$

Lemma 5.3. *Let X be a finite dimensional CAT(0) cube complex and $Y \subset X$ the essential core of an action $\Gamma \rightarrow \text{Aut}(X)$. If c is the cocycle defined in (3.12) and (3.13) and if $n \geq 2$, then $c|_{(\partial Y)^{[3]}}$ does not vanish.*

Proof. By Lemma 2.17 we may fix $(\xi_1, \xi_2, \xi_3) \in (\partial Y)^{[3]}$ with $h_1, h_2 = \gamma h_1, h_3 = \gamma' h_1$ an n -disjoint facing triple of half-spaces and $\xi_i \in \partial h_i$, for $i = 1, 2, 3$.

Because of Lemma 3.4 it is sufficient to show that the set $[[\xi_1, \xi_2]] \setminus ([[\xi_1, \xi_3]] \cup [[\xi_3, \xi_2]])$ is not empty. Let $k_1 := h_1^* \supset k_2 \supset \cdots \supset k_\ell =: h_2$ be a maximal sequence of tightly nested half spaces. By assumption, we know that $\ell \geq n$. Since h_1, h_2 and h_3 are pairwise facing then $\xi_3 \in h_1^*$ and $\xi_3 \notin h_2$. It follows that there exists $1 \leq i_0 < \ell$ such that $\xi_3 \in k_{i_0}$ and $\xi_3 \notin k_{i_0+1}$. Therefore any tightly nested subsequence of length n that contains k_{i_0} and k_{i_0+1} has the property that $s \in [[\xi_1, \xi_2]] \setminus ([[\xi_1, \xi_3]] \cup [[\xi_3, \xi_2]])$. \clubsuit

Remark 5.4. The proof of Lemma 5.3 shows that $c|_{A_h}$ does not vanish.

Proof of Theorem 1.1. According to [BI02], the class of $c \circ \varphi$ is the isometric image of the median class \mathfrak{m}_n under the isomorphism (3.15), where $\varphi : B \rightarrow \partial Y$ is the boundary map constructed in Theorem 4.1. Proposition 5.1 and Lemma 5.3 ensure that $c \circ \varphi$ is non-trivial if $n \geq 2$. \clubsuit

6. APPLICATIONS

6.1. Rigidity of Actions.

Proof of Theorem 1.3. We will need our action to satisfy the property that the barycenter of every face has trivial stabilizer. This is a natural generalization to CAT(0) cube complexes of the notion of no edge inversions in the context of actions on trees. For this reason, we start with an arbitrary action and then pass to its barycentric subdivision.

Let Y' be the barycentric subdivision of Y . Observe that $\text{Aut}(Y) \hookrightarrow \text{Aut}(Y')$ and the image acts with the property that the barycenter of every face in Y' has trivial stabilizer. Moreover Γ acts essentially on Y' and Y' is irreducible.

The proof follows very closely the strategy of the proof in [Sha00]. Namely, if we denote by e_i the identity in G_i , we aim to show that there is an $i \in \{1, \dots, \ell\}$ for which the set

$$Y_i := \{x \in Y' : \text{if } \gamma_m \in \Gamma \text{ such that } \text{pr}_i(\gamma_m) \rightarrow e_i, \\ \text{then there exists } N > 0 \text{ such that } \gamma_m x = x \text{ for all } m \geq N\}$$

is not empty, where $\text{pr}_i : G \rightarrow G_i$ is the i -th projection. It is easy to see that the set Y_i is Γ -invariant. Indeed, if $\gamma_m \in \Gamma$ is a sequence such that $\text{pr}_i(\gamma_m) \rightarrow e_i$, then for every $\gamma \in \Gamma$ we have that $\text{pr}_i(\gamma^{-1}\gamma_m\gamma) \rightarrow e_i$. Moreover Y_i is convex with respect to the CAT(0) metric: in fact, let $x_1, x_2 \in Y_i$ and let $\gamma_m \in \Gamma$ a sequence such that $\text{pr}_i(\gamma_m) \rightarrow e_i$. Then, by definition of Y_i there exists N sufficiently large such that $\gamma_m x_j = x_j$ for all $m \geq N$ and $j = 1, 2$. Since Γ acts by isometries, if $m \geq N$ then γ_m will also fix pointwise the unique CAT(0) geodesic between x_1 and x_2 .

We claim now that, if Y_i is not empty, then it is in fact a subcomplex of Y' . To see this, let us write Y' as the disjoint union of k -dimensional faces, where a k -dimensional face is the interior of a k -dimensional cube if $1 \leq k \leq \dim(Y)$ and is the boundary of a 1-dimensional cube if $k = 0$. Let F_k be a k -dimensional face which has non-empty intersection with Y_i . Then $F_k \subset Y_i$. Since we are acting on the barycentric subdivision Y' we have also that if γ_m eventually fixes a face F_k , then it fixes all lower dimensional faces that are contained in its closure $\overline{F_k}$, thus showing that $\overline{F_k} \subset Y_i$. Thus Y_i is a CAT(0) cube subcomplex of Y' .

We are then left to show that there exists $i \in \{1, \dots, \ell\}$ such that $Y_i \neq \emptyset$.

Let $i \in \{1, \dots, \ell\}$ and let $\mathcal{H}_i \subset \ell^2(\mathfrak{H}(Y)^{(n)})$ be the (possibly trivial) subspace on which the isometric action of Γ extends continuously to G via the projection $\text{pr}_i : G \rightarrow G_i$. By [BM02, Theorem 16] there exists an $i \in \{1, \dots, \ell\}$ (not necessarily unique) such that $\mathcal{H}_i \neq \{0\}$. We fix this i .

The space Y_i will be constructed from this data as follows.

Define on $\mathfrak{H}(Y)^{(n)}$ an equivalence relation, namely if $s, s' \in \mathfrak{H}(Y)^{(n)}$, we say that $s \sim s'$ if $f(s) = f(s')$ for all $f \in \mathcal{H}_i$. Since these functions are square summable, all of the equivalence classes are finite, with the possible exception of the class where all functions in \mathcal{H}_i vanish. Moreover Γ permutes all the finite equivalence classes and leaves invariant the only infinite one. Therefore, the complement of the infinite class in $\mathfrak{H}(Y)^{(n)}$, which we denote by $\mathfrak{H}(Y)_0^{(n)}$ is Γ -invariant.

Claim 6.1. *Let $[s] \in \mathfrak{H}(Y)_0^{(n)} / \sim$ be a finite equivalence class and $\text{Stab}_\Gamma([s])$ its stabilizer. If $\gamma_m \in \Gamma$ is a sequence such that $\text{pr}_i(\gamma_m) \rightarrow e_i$, then there exists $N > 0$ such that for all $m \geq N$, $\gamma_m \in \text{Stab}_\Gamma([s])$.*

We assume the claim for the moment and show that Y_i is not empty. Fix $[s] \in \mathfrak{H}(Y)_0^{(n)}$ and let \hat{h} be a hyperplane corresponding to one of the half spaces appearing in an element of $[s]$. By Lemma 2.6, there exists $\gamma \in \Gamma$ such that \hat{h} and $\gamma\hat{h}$ are strongly separated. By [BC12, Lemma 2.2], the bridge $b(\hat{h}, \gamma\hat{h})$ – that is the set of minimal length geodesics connecting \hat{h} and $\gamma\hat{h}$ – consists of a single geodesic (of finite length). Observe that, since $\mathfrak{H}(X)_0^{(n)}$

is invariant, also the class $[\gamma s]$ is finite. Hence there are finitely many half spaces in the set $\{h : h \in s', \text{ for } s' \in [s']\}$, both if $s' = s$ and if $s' = \gamma s$. It follows that if we define $L := \text{Stab}_\Gamma([s]) \cap \text{Stab}_\Gamma([\gamma s])$, then the L -orbit of $b(\hat{h}, \gamma \hat{h})$ is finite, therefore bounded, and its circumcenter is an L -fixed point.

If $\gamma_m \in \Gamma$ is a sequence such that $\text{pr}_i(\gamma_m) \rightarrow e_i$, Claim 6.1 implies that, for m large enough, the sequence γ_m is in L and hence fix the circumcenter, thus showing that $Y_i \neq \emptyset$.

Since the action of Γ on Y' is essential, then $Y_i = Y'$. Observe however that, since $\text{Aut}(Y)$ is closed in $\text{Aut}(Y')$ in the topology of the pointwise convergence, then the extension of the action to G is in $\text{Aut}(Y)$. \clubsuit

Proof of Claim 6.1. Let $s \in \mathfrak{H}(Y)_0^{(n)}$ and let $f \in \mathcal{H}_i$ so that $f(s) \neq 0$. Since $\lim_{m \rightarrow \infty} \|\gamma_m f - f\|_2 = 0$, then $\lim_{m \rightarrow \infty} f(\gamma_m s) = f(s)$. Because f is square integrable, it takes finitely many values in a $|f(s)|/2$ -neighborhood of $f(s)$, so that we conclude that there exists $N(f, s)$ such that $f(\gamma_m s) = f(s)$ for all $m \geq N(f, s)$. In particular $\{\gamma_m s : m \geq 1\}$ is finite.

If $\gamma_{m_k} s \not\sim s$ for some subsequence m_k , then, by passing to a further subsequence, we may assume that $s_0 := \gamma_{m_k} s \not\sim s$. But then there is $g \in \mathcal{H}_i$ such that $g(s) \neq 0$ and $g(s_0) \neq g(s)$, which, together with

$$g(s_0) = \lim_k g(\gamma_{m_k} s) = g(s),$$

is a contradiction. \clubsuit

The proof of the above theorem does rely on the assumption that Y is irreducible and essential. In general, we can pass to the essential core Y of the Γ -action on X and to its barycentric subdivision Y' . Let $\Gamma' < \Gamma$ be the finite index subgroup that acts on each of the irreducible factors in Y' and let $G'_i := \overline{\text{pr}_i(\Gamma')}$. By applying Theorem 1.3 to each of the irreducible factors we obtain that the action of Γ' on Y extends continuously to an action of G' , where $G' = G'_1 \times \cdots \times G'_\ell$, by factoring via one of the factors. We have hence proven the following:

Corollary 6.2. *Let X be a finite dimensional $\text{CAT}(0)$ cube complex and Γ be an irreducible lattice in the product of locally compact groups $G_1 \times \cdots \times G_\ell =: G$. Let $\Gamma \rightarrow \text{Aut}(X)$ be a non-elementary action on X . Then the action of Γ on the essential core of X virtually extends to a continuous action of an open finite index subgroup in G , by factoring via one of the factors.*

6.2. The Class \mathcal{C}_{reg} . We now prove Corollary 1.6 concerning the class of groups \mathcal{C}_{reg} . The idea of the proof is as follows. If the action is proper, then in particular the vertex stabilizers are finite. We can then find, for n sufficiently large, elements $s \in \mathfrak{H}(X)^{(n)}$ with finite stabilizers such that $\ell^p(\Gamma \cdot s) \hookrightarrow \ell^p(\Gamma)$. We then prove that s can be chosen in such a way that this map does not vanish on the image of the cocycle.

Proof of Corollary 1.6. We start with the observation that if $s \in \mathfrak{H}^{(m)}$, with $m \geq 2$, such that $\text{Stab}_\Gamma(s)$ is finite, then there is Γ -equivariant map

$$(6.25) \quad \sigma_s : \ell^p(\Gamma \cdot s) \rightarrow \ell^p(\Gamma)$$

defined by $\sigma_s f(\gamma) := f(\gamma s)$. Since $\|\sigma_s f\|_p = |\text{Stab}_\Gamma(s)| \|f\|_p$, the map is injective. We hence look for sequences with finite stabilizer for some $m \geq 2$.

We begin by noticing that if $h \subset k$ are two strongly separated half spaces in \mathfrak{H} , then, there is a finite CAT(0) bridge between them [BC12]. Since the action of Γ is also by CAT(0)-isometries, the stabilizer of $\{h, k\}$ must also stabilize their CAT(0) bridge. Since the action is proper, the stabilizer of the CAT(0) bridge $b(\hat{h}, \hat{k})$ is finite. It follows that if s is any tightly nested sequence such that two of its half spaces are strongly separated, then s has finite stabilizer as well. Thus it is enough to find s that contains pairs of strongly separated half spaces.

Recall that in Lemma 2.17 for every n and every $\hat{h} \in \hat{\mathfrak{H}}$ we found $\gamma, \gamma' \in \Gamma$ such that $\hat{h}, \gamma\hat{h}, \gamma'\hat{h}$ are an n -disjoint facing triple of pairwise strongly separated hyperplanes. Let us consider the finite set $S(\hat{h}, \gamma\hat{h}, \gamma'\hat{h})$ of tightly nested sequences that contain one pair of strongly separated half spaces among the half spaces corresponding to $\hat{h}, \gamma\hat{h}, \gamma'\hat{h}$. Then any sequence $s \in S(\hat{h}, \gamma\hat{h}, \gamma'\hat{h})$ satisfies the requirement that $|\text{Stab}_\Gamma(s)| < \infty$. Let ℓ be the maximal length of a sequence in $S(\hat{h}, \gamma\hat{h}, \gamma'\hat{h})$ and let $S'(\hat{h}, \gamma\hat{h}, \gamma'\hat{h})$ be the set of sequences in $S(\hat{h}, \gamma\hat{h}, \gamma'\hat{h})$ "completed" to have all length ℓ . The stabilizer of a sequence in $S'(\hat{h}, \gamma\hat{h}, \gamma'\hat{h})$ is still finite.

It follows that $c|_{A_h}$ takes values into $\bigoplus_{s \in S'(\hat{h}, \gamma\hat{h}, \gamma'\hat{h})} \ell^p(\Gamma \cdot s) \subset \ell^p(\mathfrak{H}(Y)^{(\ell)})$. By Remark 5.4, $c|_{A_h}$ does not vanish, hence at least one of the projections on a direct summand, say $\ell^p(\Gamma \cdot s_0)$, must be in the essential range of $c|_{A_h}$. Then the image of its composition with such projection followed by σ_{s_0} in (6.25), with $m = \ell$, is in $\ell^p(\Gamma)$. Setting $S_h := \varphi^{-1}(h) \times \varphi^{-1}(\partial(\gamma h)) \times \varphi^{-1}(\partial(\gamma' h))$ and using Proposition 5.1 with $E = \ell^p(\Gamma)$, we conclude that the class $c \circ \varphi|_{S_h}$ does not vanish and hence, again by (3.15), $H_b^2(\Gamma, \ell^p(\Gamma)) \neq 0$. \ast

APPENDIX A. SOME MORE PROOFS

A.1. The Measurability of Certain Key Maps. The notation used in this section refers to § 4.

Lemma A.1. *Let $I \subseteq [0, 1]$ be a subinterval, possibly open, half open, or closed. Let $H_\mu^I = \{h \in \mathfrak{H}(X) : \mu(h) \in I\}$. The map $\mathcal{P}(\overline{X}) \rightarrow 2^{\mathfrak{H}(X)}$, defined by $\mu \mapsto H_\mu^I$ is measurable with respect to the weak- \ast topology on $\mathcal{P}(\overline{X})$.*

As a consequence, the map $N : \mathcal{P}(\overline{X}) \rightarrow \mathbb{N} \cup \{\infty\}$ defined by $N(\mu) = |H_\mu^I|$ is measurable.

Proof. Recall the definition of cylinder sets: Let $F_1, F_2 \in 2^{\mathfrak{H}(X)}$ be two finite sets. The cylinder set associated to them is

$$C(F_1, F_2) = \{H \in 2^{\mathfrak{H}} : F_1 \subseteq H \text{ and } F_2 \subseteq H^c\}.$$

Cylinder sets form a basis for the topology on $2^{\mathfrak{H}(X)}$. Therefore it is sufficient to show that

$$K(F_1, F_2) = \{\mu : H_\mu^I \in C(F_1, F_2)\}$$

is measurable.

To this end, observe that h is open and closed as a subset of \overline{X} so that its characteristic function $\mathbb{1}_h$ is continuous. Therefore, the set $E_I(h) = \{\mu : \mu(h) \in I\}$, for $h \in \mathfrak{H}(X)$ is weak-* open, half open, or closed according to I and therefore is measurable. Then the following finishes the proof

$$K(F_1, F_2) = \bigcap_{h \in F_1} E_I(h) \cap \bigcap_{h \in F_2} (E_I(h))^c.$$

✱

Corollary A.2. *The following maps are measurable:*

- (1) $C_1 : \mathcal{P}(\overline{X}) \rightarrow \mathbb{N} \cup \{\infty\}$, defined by $C_1(\mu) := |H_\mu|$;
- (2) $C_2 : \mathcal{P}(\overline{X}) \times \mathcal{P}(\overline{X}) \rightarrow \mathbb{N} \cup \{\infty\}$, defined by $C_2(\mu, \nu) := |H_\mu \cap H_\nu|$;
- (3) $T : \mathcal{P}(\overline{X}) \times \mathcal{P}(\overline{X}) \rightarrow \mathbb{N} \cup \{\infty\}$, defined by
$$T(\mu, \nu) := |\tau((H_\mu \cap H_\nu^+) \cup (H_\nu \cap H_\mu^+))|;$$

- (4) $N_\nu : \mathcal{P}(\overline{X}) \rightarrow \mathbb{N} \cup \{\infty\}$, defined by $N_\nu(\mu) := |(H_\mu \times H_\nu) \cap \mathcal{S}|$, where
$$\mathcal{S} := \{(h_1, h_2) \in \mathfrak{H}(X) \times \mathfrak{H}(X) : h_1, h_2 \text{ are strongly separated}\}.$$

- (5) $C_3 : \mathcal{P}(\overline{X}) \times \mathcal{P}(\overline{X}) \rightarrow \mathbb{N} \cup \{\infty\}$, defined by
$$C_3(\mu, \nu) := |[H_\mu^+ \cap (H_\nu \setminus H_\mu)] \cup [H_\nu^+ \cap (H_\mu \setminus H_\nu)]|.$$

Proof. The proofs of statements (1), (2), and (5) are consequences of Lemma A.1 and the observation that the product and composition of Γ -equivariant measurable maps is again Γ -equivariant and measurable. Statement (4) follows by considering the fact that \mathcal{S} is a Γ -invariant set. Statement (3) follows from the following result. ✱

Lemma A.3. *The map $p : 2^{\mathfrak{H}(X)} \rightarrow 2^X$ defined by $\mathbb{1}_S \mapsto \mathbb{1}_{\bigcap_{h \in S} h}$ is measurable.*

Proof. Choose an enumeration $\mathfrak{H}(X) = \{h_n : n \in \mathbb{N}\}$. Recall that the standard projection $\pi_N : 2^{\mathfrak{H}(X)} \rightarrow 2^{\{h_1, \dots, h_N\}}$ is continuous. Next define $p_N : 2^{\{h_1, \dots, h_N\}} \rightarrow 2^X$ as $\mathbb{1}_S \mapsto \mathbb{1}_{(\bigcap_{h \in S} h)}$ which is also continuous as $2^{\{h_1, \dots, h_N\}}$ is endowed with the discrete topology.

Observe that $p(\mathbb{1}_S) = \sup\{p_N \circ \pi_N(\mathbb{1}_S) : n \in \mathbb{N}\}$ and is hence measurable as the supremum of continuous functions. \clubsuit

Recall that in (3.10) we set the notation $\mathcal{E}_p := \ell^q(\mathfrak{H}(X)^{(n)})$ if $1/p + 1/q = 1$ and $1 < p < \infty$, and \mathcal{E}_1 to be the Banach space of functions on $\mathfrak{H}(X)^{(n)}$ that vanish at infinity.

Lemma A.4. *for all $1 \leq p < \infty$, the cocycle $c : \overline{X}^3 \rightarrow \ell^p(\mathfrak{H}^{(n)})$ is a Borel map, where $\overline{X}^3 \subset 2^{\mathfrak{H}(X)}$ has the induced product topology and $\ell^p(\mathfrak{H}(X)^{(n)})$ has the weak-* topology as the dual of \mathcal{E}_p .*

Proof. Choose an enumeration of \mathfrak{H} and let $\mathfrak{H}_N := \{h_1, \dots, h_N\}$. Let us define the finite space

$$\mathfrak{H}_N^{(n)} := \{s \in \mathfrak{H}^{(n)} : s \subset \mathfrak{H}_N\},$$

and, for any subsets $E, F \subset \mathfrak{H}$, the set

$$(E \setminus F)_N^{(n)} := \{s \in \mathfrak{H}_N^{(n)} : s \subset E \text{ and } s \not\subset F\}.$$

The map $c_N^+ : 2^{\mathfrak{H}} \times 2^{\mathfrak{H}} \times 2^{\mathfrak{H}} \rightarrow C_0(\mathfrak{H}^{(n)})$ defined as

$$c_N^+(F_1, F_2, F_3) := \mathbb{1}_{(F_3 \setminus F_2)_N^{(n)}} + \mathbb{1}_{(F_1 \setminus F_3)_N^{(n)}} + \mathbb{1}_{(F_2 \setminus F_1)_N^{(n)}}$$

factors through the canonical projection $2^{\mathfrak{H}} \rightarrow 2^{\mathfrak{H}_N}$ on triples and hence is continuous. Then the map $c_N(F_1, F_2, F_3) := c_N^+(F_1, F_2, F_3) - c_N^+(F_1, F_3, F_2)$ is also continuous and in particular is continuous when restricted to the subset $\overline{X}^3 \subset (2^{\mathfrak{H}})^3$. (Here we use the identification of a vertex $v \in X$ with the principal ultrafilter containing v .)

For any $f \in \mathcal{E}_p$, the function on \overline{X}^3 defined by

$$(x, y, z) \mapsto \langle c_N(x, y, z), f \rangle$$

is continuous. Its pointwise limit

$$(x, y, z) \mapsto \lim_{N \rightarrow \infty} \langle c_N(x, y, z), f \rangle$$

is measurable and, in fact,

$$\langle c(x, y, z), f \rangle = \lim_{N \rightarrow \infty} \langle c_N(x, y, z), f \rangle,$$

thus showing that the cocycle c restricted to \overline{X}^3 is Borel. \clubsuit

A.2. A Lemma in Graph Theory. Let $\mathcal{G}(V, E)$ a complete directed finite graph⁶ with vertices V and edges E . We denote by $s, t : E \rightarrow V$ respectively the *source* and the *target* of an edge. We allow the possibility that there are two edges between two vertices, one in each direction. Given a vertex $v \in V$, we denote by $o(v)$ (respectively $i(v)$) the number of outgoing (respectively incoming) edges at v . Since the graph is complete,

$$(1.26) \quad o(v) + i(v) \geq |V| - 1,$$

for every $v \in V$.

The next lemma shows that if the graph is complete, there is at least one vertex that has “many” outgoing edges.

Lemma A.5. *If $\mathcal{G} := \mathcal{G}(V, E)$ is a complete directed finite graph and $|V| = D$, then there exists $v \in V$ such that $o(v) \geq \frac{D-1}{2}$.*

Proof. From (1.26) we have that

$$\sum_{v \in V} o(v) + i(v) \geq D(D-1).$$

We have also that

$$\sum_{v \in V} o(v) = \sum_{v \in V} i(v),$$

so that

$$\sum_{v \in V} o(v) \geq \frac{D(D-1)}{2}.$$

Since $|V| = D$, the assertion follows readily. ✱

Definition A.6. We say that a complete directed finite graph $\mathcal{G}(V, E)$ with $|V| = D$ is *strictly upper triangular*⁷ if there exists a numbering v_1, \dots, v_D of its vertices, such that for all $j = 1, \dots, D$,

$$\begin{aligned} o(v_j) &= D - j \\ i(v_j) &= j - 1 \end{aligned}$$

The terminology is inspired from the fact that the corresponding $D \times D$ adjacency matrix M with coefficients

$$M_{ij} := \begin{cases} 1 & \text{if there exists } e \in E \text{ with } s(e) = v_i \text{ and } t(e) = v_j \\ 0 & \text{otherwise,} \end{cases}$$

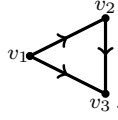
is strictly upper triangular, namely v_1 is connected by an outgoing vertex to all of the remaining v_2, \dots, v_D , v_2 is connected to v_3, \dots, v_D and so on.

Example A.7. A strictly upper triangular graph with $d = 2$ corresponds to the matrix

$$M = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \text{ and is of the form}$$

⁶In graph theory a graph with these properties is called *tournament*.

⁷Or *transitive tournament*.



Lemma A.8. *Let $\mathcal{G} = \mathcal{G}(V, E)$ be a complete directed graph (not necessarily finite) and $D \in \mathbb{N}$. If $|V| \geq 5^D$, there exist D vertices v_1, \dots, v_D such that the induced complete directed subgraph on v_1, \dots, v_D is strictly upper triangular.*

Proof. The idea is to construct the strictly upper triangular graph inductively. By Lemma A.5 there exists $v_1 \in V$ with $o(v_1) \geq \frac{|V|-1}{2} \geq \frac{|V|}{5}$ outgoing edges. Denote by

$$O(v_1) := \{e \in E : s(e) = v_1\}$$

the set of outgoing edges (so that $|O(v_1)| = o(v_1)$). We consider now the induced complete directed subgraph $\mathcal{G}(v_1)$ on v_1 and on the vertices at the end of the edges in $O(v_1)$, namely $\mathcal{G}_1 := \mathcal{G}(V(v_1), E(v_1)) \subset \mathcal{G}$, where

$$V(v_1) := \{v_1\} \sqcup \{t(e) : e \in O(v_1)\} \quad \text{and} \quad O(v_1) \subsetneq E(v_1) \subseteq E,$$

and the edges $E(v_1)$ are exactly the edges in E needed to make complete the graph on the vertices $V(v_1)$. Remark that, by construction, $|V(v_1)| = o(v_1) + 1 \geq \frac{|V|}{5} \geq 5^{D-1}$ and the induced complete directed subgraph on any ordered pair of vertices v_1, v , with $v \in V(v_1)$, is (trivially) a strictly upper triangular graph.

We could now proceed to formulate a rigorous proof by induction, but we prefer showing how to move to the next step, as we believe that the very simple idea of the proof will be more transparent.

We repeat now exactly the same construction as before, applied to the graph $\mathcal{G}_1(V(v_1), E(v_1))$ instead of $\mathcal{G}(V, E)$. Namely, let $v_2 \in V(v_1)$ be the vertex, whose existence is asserted by Lemma A.5, such that if

$$O(v_2) := \{e \in E(v_1) : s(e) = v_2\},$$

then $o(v_2) = |O(v_2)| \geq \frac{|V(v_1)|-1}{2} \geq \frac{|V(v_1)|}{5}$. By construction there is an outgoing edge from v_1 to v_2 and from v_1 to any other vertex in \mathcal{G}_1 . From the graph \mathcal{G}_1 we retain now only those vertices $w \in V(v_1)$ that are at the end of an outgoing edge from v_2 , so that v_1, v_2, w is strictly upper triangular, and eliminate all of the other vertices. Namely, let \mathcal{G}_2 be the induced complete directed graph on v_2 and on the vertices that are the targets of the $o(v_2)$ edges in $E(v_1)$ outgoing from v_2 . That is $\mathcal{G}_2 := \mathcal{G}(V(v_2), E(v_2)) \subset \mathcal{G}_1$, where

$$V(v_2) := \{v_2\} \sqcup \{t(e) : e \in O(v_2)\} \quad \text{and} \quad O(v_2) \subsetneq E(v_2) \subseteq E(v_1) \subseteq E.$$

Now we have a complete directed graph on $|V(v_2)| \geq o(v_2) + 1 \geq \frac{|V(v_1)|}{5} \geq 5^{D-2}$ vertices from which we can continue choosing vertices v_3, \dots, v_D such that at each step we increase by one our strictly upper triangular graph and we reduce by a factor of 5 the cardinality of the vertex set. ✱

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UNIVERSITÉ D'ORLÉANS, ORLÉANS, FRANCE

E-mail address: `indira.chatterji@univ-orleans.fr`

UNIVERSITY OF NORTH CAROLINA AT GREENSBORO

E-mail address: `t_fernos@uncg.edu`

DEPARTMENT MATHEMATIK, ETH, CH-8092 ZÜRICH, SWITZERLAND

E-mail address: `iozzi@math.ethz.ch`